Cognitive Hierarchies in the Minimizer Game

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Abstract

Experimental tests of choice predictions in one-shot games show only little support for Nash equilibrium (NE). Poisson Cognitive Hierarchy (PCH) and level-k (LK) are behavioral models of the thinking-steps variety where subjects differ in the number of levels of iterated reasoning they perform. Camerer et al. (2004) claim that substituting the Poisson parameter $\tau = 1.5$ yields a parameter-free PCH model (pfPCH) which predicts experimental data considerably better than NE. We design a new multi-person game, the Minimizer Game, as a testbed to compare initial choice predictions of NE, pfPCH and LK. Data obtained from two large-scale online experiments strongly reject NE and LK, but are well in line with the point-prediction of pfPCH.

JEL classification: C72; C90; D01; D83

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1 Introduction

1.1 Nash equilibrium in one-shot games

Nash equilibrium (NE) is the central solution concept in noncooperative game theory, but it is well known that NE imposes extremely demanding assumptions on players’ rationality as well as on the consistency of their beliefs (Aumann and Brandenburger, 1995). Experimental tests of game theoretic strategy choice predictions in one-shot games, where players are inexperienced and have no possibility of learning, have consistently shown only little support for NE except in rather specific games (Camerer, 2003). Game theorists have therefore worked out a variety of alternative explanations and models for prediction of choices in one-shot games. One such approach is to keep the rationality assumption and weaken the mutual consistency requirements. Any concept following this approach must address the question of how players form their beliefs about other players’ actions. This has led to the formulation of so-called thinking steps models.

1.2 Level-\( k \) models

The basic assumption of the thinking steps approach is that players differ in the number of steps of iterated reasoning they apply when deliberating which strategy they should pick in a strategic choice problem. Nagel (1995) used a simple thinking steps model to explain the results of her experiments about number choices in the \( p \)-beauty-contest game (or \( p \)-guessing game), where players choose a number from \([0, 100]\) and whoever comes closest to \( p \) times the average of the chosen numbers wins a fixed prize. Her model explains the “spikes” around choices of 33 and 22, which are often observed in experimental data for the typical parameter \( p = 2/3 \). Similar thinking steps models have been proposed by Stahl and Wilson (1994, 1995) and by Ho et al. (1998).
The most prominent model of the thinking steps variety is the level-$k$ model (LK model) introduced by Costa-Gomes et al. (2001). It proposes that most players can be classified as level-$k$ ($l_k$) types, which anchor their beliefs in an $l_0$ type who does not think strategically at all but just chooses from a uniform random distribution on the set of pure strategies.$^1$ $l_k$ then simply best responds (possibly with noise) to $l(k - 1)$. These $l_k$ types are complemented by types $d_k$, who best respond to a uniform distribution of beliefs on strategies surviving $k$ rounds of iterated dominance, respectively. Finally, some players might be equilibrium types, choosing an equilibrium strategy, or sophisticated, best responding to an accurate distribution of beliefs on other types. LK models have been applied in econometric analyses of various experimental data by Costa-Gomes and Crawford (2006), Crawford and Iriberri (2007a, 2007b), and Costa-Gomes et al. (2009), among others. These studies spawned a large body of literature (see the recent review by Crawford et al., 2013). By and large, the common view that emerged from this literature is that $l_1$ and $l_2$ types are predominant in subject populations, complemented by smaller fractions of $l_3$ and possibly $l_4$, $d_1$ and equilibrium types. $l_0$ as well as $l_5$ or higher, $d_2$ or higher, and sophisticated types, however, are virtually absent from the population.

### 1.3 The Poisson Cognitive Hierarchy model

A closely related thinking steps model is the Poisson Cognitive Hierarchy model (PCH model) of Camerer et al. (2004). This model uses only a single parameter, $\tau$. It is based on the view that players differ in their level $k$ of iterated thinking, and that $k$ is distributed in the population of players following

$^1$A vast majority of applications uses this specification of $l_0$ behavior. Alternatively $l_0$ has also been suggested to choose the most salient strategy in games with non-neutral frames (Crawford and Iriberri, 2007b), but this approach is not without problems itself (Hargreaves Heap et al., 2014). Burchardi and Penczynski (2014) use an innovative experimental design to identify $l_0$ reasoning in beauty-contest games. For a recent systematic approach to $l_0$ behavior see Wright and Leyton-Brown (2014a).
a Poisson distribution with mean (and therefore variance) $\tau$. Moreover, while a level-$k$ ($L_k$) player thinks that all other players do less steps of reasoning than he himself, he is aware of the presence of all levels of reasoning from 0 to $k - 1$ in the population. The frequency he believes these lower levels to occur are the true (Poisson) frequencies, truncated at $k - 1$ and normalized so as to add up to 1.

The PCH model has been shown to predict reasonably well in a variety of games, among them $p$-beauty-contest games with $p < \frac{1}{2}$, market entry games, $3 \times 3$ bimatrix games (Camerer et al., 2004), coordination games (Costa-Gomes et al., 2009), and the action commitment game (Carvalho and Santos-Pinto, 2014). While, as expected, the best-fitting value of $\tau$ is game- and population-specific, Camerer et al. (2004) report that a value of $\tau = 1.5$, corresponding to a population dominated by $L1$ and $L2$ types, is able to explain experimental data considerably better than Nash equilibrium across a variety of experimental games.

While the PCH model is simple to apply and has proven useful in a number of games, it also seems to fail in some specific classes of games. For example, it is well known that in Prisoner’s Dilemma and Public Goods experiments initial cooperation levels are substantial (see Camerer, 2003). Under the assumption of self-regarding preferences such behavior cannot be explained by a PCH model, since there all types but $L0$ optimize and hence never choose dominated strategies. For the same reason, PCH cannot account for the puzzling majority choices of dominated strategies in the two-person beauty contest of Grosskopf and Nagel (2008). As Camerer et al. (2004) report, the PCH model also predicts almost random choice in $p$-beauty-contest games with $p > 1$. Another problem arises in games with large strategy spaces. Camerer et al. (2002) mention that in such games PCH predicts only a small fraction of the strategies actually chosen. An example for this

\footnote{There is, however, some discussion about how well the PCH model really predicts in these games, see e.g. Hahn et al. (2010) or chapter 17.3 of Moffatt (2015).}
is found in Gneezy (2005), where prior to grouping the data PCH cannot account for bid choices in a first-price auction with 100 pure strategies.

Camerer et al. (2002) devote section 4.1 of their working paper to investigate what went wrong in games 2, 6, and 8 of the 12 games of Stahl and Wilson (1995). These are symmetric $3 \times 3$-games where the best-fitting PCH model is $\tau = 0$, predicting purely random choice. But actually roughly half of the subjects picked their Nash equilibrium strategy in these three games. Camerer et al. (2002) speculate that the experimental procedure of Stahl and Wilson may have catalyzed a large fraction of Nash play. However, a more parsimonious explanation derives from the observation that in these three games the Nash equilibrium strategy also happened to be the unique maximin choice. Maximin choices actually predict the majority choices in ten out of the 12 Stahl-Wilson games. In games where strategic thinking is cognitively demanding, the nonstrategic and risk-averse maximin choice is an easy option and might often be a more salient anchor than uniform randomization for level-0 types (see also Van Huyck et al., 1991).

1.4 Predicting choice with the parameter-free Poisson Cognitive Hierarchy model

As noted by Wright and Leyton-Brown (2014b), the bulk of the literature on thinking steps models is concerned more with explaining than with predicting behavior. Typically, type distributions and other parameters are estimated from experimental training data while direct prediction performance comparisons are rare. In this paper we focus on prediction in a very strict sense. What we aim at is the prediction of initial choices without any prior parameter estimation. For this we need a parameter-free model which can be directly pitted against Nash equilibrium.

To our knowledge, within the thinking steps variety the only parameter-free model to be found in the literature is the PCH model which results from substituting $\tau = 1.5$ for the Poisson parameter, suggested by Camerer et
al. (2004). In their Abstract they state that \textit{an average of 1.5 steps fits data from many games} (p. 861); they note that \textit{values of }$\tau$\textit{ between 1 and 2 explain empirical results for nearly 100 games, suggesting that assuming a }$\tau$\textit{ value of 1.5 could give reliable predictions for many other games as well} (p. 863) and that the data \textit{suggest that the Poisson-CH model with }$\tau = 1.5$\textit{ can be used to reliably predict behaviors in new games} (p. 877). In their conclusion, Camerer et al. (2004) stress that \textit{the value }$\tau = 1.5$\textit{ is a good omnibus guess which makes the Poisson-CH theory parameter-free and is very likely to predict as accurately as Nash equilibrium, or more accurately, in one-shot games} (p. 890).

We call the PCH model with $\tau = 1.5$ the \textit{parameter-free PCH model} (pf-PCH model). The pfPCH model states that the population wide frequencies of levels $L_0$, $L_1$, $L_2$, and $L_3$ are given by 22.3\%, 33.5\%, 25.1\%, and 12.6\%, respectively, with only 6.6\% accruing from levels 4 and higher. The value $\tau = 1.5$ for the mean (and the variance) of the number of thinking-steps is based on experimental data from various games scrutinized by Camerer et al. (2004). Does this value also predict reasonably out of sample, i.e. in games beyond the classes of games it was derived from? To evaluate the predictive performance of the pfPCH model we use it to predict the distribution of initial choices in a new game, the \textit{minimizer game} described below, which we motivate and construct specifically for this purpose.

We do not only compare the pfPCH-prediction to the Nash-prediction, but also to the LK model’s prediction. The LK model is not parameter-free and we are not aware of any suggestions for a “good omnibus guess” of LK type frequencies from the literature. We therefore take a generous approach and allow for \textit{all} type distributions of the LK model. We find that in our minimizer game the pfPCH-prediction easily outperforms both Nash equilibrium and all LK model specifications.
2 The Minimizer Game

How well does the pfPCH model predict players’ initial response to a strategic choice problem? We planned to answer this question experimentally and started by asking what kind of game would be appropriate for an experimental test of this question. In our opinion, three issues had to be considered:

- Experimental outcomes in Prisoner’s Dilemma games, Public Goods games, Dictator games, Ultimatum games and the like are strongly influenced by the presence of altruistic motives, fairness considerations, or other social preferences. These preferences “contaminate” the experimental results, since they may override the strategic incentives created by the monetary payoffs.3 We should therefore avoid games where choices are sensitive to the presence of social preferences. This basically rules out almost all two-player games. It seems therefore wise to look for a multi-person game where other-regarding preferences are unlikely to influence choices.

- Since the premise of the thinking steps models is that given their beliefs, players optimize, we should choose a simple game for our test. If due to computational complexity subjects get the arithmetic wrong when optimizing, choices will be biased even if the PCH model accurately describes belief formation and the distribution of thinking steps. It is to a large extent a matter of taste what kind of game to deem simple. However, bearing in mind Grosskopf and Nagel’s (2008) stunning results for two-person beauty contest games, where even a majority of professionals failed to realize that 0 is a dominant strategy, we would strongly opt for avoidance of the need of any arithmetic having to be done by subjects trying to optimize. Moreover, simplicity seems to

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3Wright and Leyton-Brown (2014a) find that the feature of fairness of an action is especially prone to influence level-0 behavior.
require a rather small set of possible choices, since with a larger number of choices both formation of beliefs and optimization given beliefs become computationally demanding.

- A third aspect to be aware of are attitudes towards risk. Strategic uncertainty may lead subjects with different risk attitudes to different choices, which could then falsely be attributed to different levels of iterated reasoning. As discussed above, nonstrategic and strongly risk-averse subjects might tend to choose their maximin strategy in the face of strategic uncertainty, again biasing the distribution of thinking steps. Ideally we would therefore construct an experimental game where maximin does not restrict the set of available choices.

2.1 Definition of the minimizer game

Considering these three points we specifically designed a game for our experiments which to the best of our knowledge has not been studied before. For reasons which will become clear in a moment, we call this game the minimizer game (MG). The formal definition of an MG is the following.

Let \( P = \{1, \ldots, I\} \) be a finite set of players, let \( N \) be a finite subset of \( \mathbb{N} \), and let \( n = |N| \) be the size of \( N \). Players’ strategy sets are \( S_i = N \) for all \( i \), so each player chooses a number from the set \( N \). For a pure strategy profile \( s = (s_i)_{i \in I} \) and for \( k \in N \) let \( c_k(s) = |\{i \in P : s_i = k\}| \) count the number of players choosing \( k \) in the profile \( s \). Let \( M(s) = \text{argmin}_{k \in N} \{c_k(s) : c_k(s) \geq 1\} \) be the set containing the numbers which have been chosen least often among those chosen at all in profile \( s \). Let \( m(s) = \min M(s) \) be the smallest of these numbers. The payoff function is identical for all players and is given by

\[
u_i(s) = m(s) \frac{1}{c_{m(s)}(s)} \text{ if } s_i = m(s) \text{ and } u_i(s) = 0 \text{ else.}
\]

Despite its technically sounding definition, it is easy to explain the MG in an extremely simple and intuitive way. The rules of the game state that each subject may choose its desired payoff from a given set of possible integer
payoffs. The ‘winning amount’ is the amount chosen least often in total - the minimizer. Among all players who chose the minimizer, one player is randomly drawn to receive this amount, while all others receive zero.\footnote{An even simpler, deterministic variant of the MG lets all players having chosen the minimizer receive this amount, i.e. $u_i(s) = m(s)$ if $s_i = m(s)$ and $u_i(s) = 0$ else. While this variant was the first we came up with, it is technically almost infeasible in our large-population online experimental approach, which is why we proceeded to work with the stochastic variant of the MG described here.} Ties are broken by declaring the smallest of the least often chosen amounts to be the minimizer.

While this is neither necessary nor specified in the definition, when it comes to experiments we implicitly think of the MG as being played with a small number of possible choices and a large number of players. The reasons for this are explained in the section on equilibrium analysis of the MG.

### 2.2 Advantages of the MG

The MG appears markedly dissimilar from strategic choice situations which are frequently encountered in everyday life. It may thus be considered artificial, but we consider this an explicit advantage. The reason is that we focus on choices in truly one-shot games, and for this we have to make it unlikely that subjects can transfer experiences from related games they “played” in the past. We think that this is the case for the MG. While the MG has many features of a congestion game,\footnote{The MG should not be confused with the superficially similar minority game, which builds upon the \textit{El Farol bar problem} (Arthur, 1994) and has been studied intensively in the statistical physics literature. It is related, but not identical to the \textit{LUPI Lottery game} (Östling et al., 2011) either. Note that in the LUPI game, contrary to the MG, the prize for the winner is independent of the winning number. Moreover, if there is no uniquely chosen number, then all payoffs are zero in the LUPI game. Thus, the LUPI game is interesting if there are many more available choices than players, while we study the MG in the exactly opposite case.} its peculiar rules make it very unlike the “typical” congestion games people unconsciously play, like e.g. choosing the fastest road to their office in the morning.

Let us now reconsider our list of three points.

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  \item [1] The MG should not be confused with the superficially similar minority game, which builds upon the \textit{El Farol bar problem} (Arthur, 1994) and has been studied intensively in the statistical physics literature. It is related, but not identical to the \textit{LUPI Lottery game} (Östling et al., 2011) either. Note that in the LUPI game, contrary to the MG, the prize for the winner is independent of the winning number. Moreover, if there is no uniquely chosen number, then all payoffs are zero in the LUPI game. Thus, the LUPI game is interesting if there are many more available choices than players, while we study the MG in the exactly opposite case.
\end{itemize}
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• Social preferences do not interfere in the MG, at least if the number of players $I$ is large. Since the influence of one’s own choice on the overall minimizer is typically negligible, there is no obvious way social preferences, even if present, could influence choice behavior.

• When it comes to optimizing, the MG is extremely simple. Given a belief about the individual distribution of number choices of other players, maximization of expected payoff is straightforward if $I$ is large: Choose the number which has the lowest frequency. As opposed to beauty contest games, bimatrix games, or first-price auction games, this does not require subjects to perform any arithmetical operations.

• In the MG, any choice might result in a zero payoff in the worst case. Therefore maximin has no bite in this game.

Taken together, these advantages indicate that the MG (with large $I$ and small $n$) is well suited to test the pfPCH model in a “purified” context, i.e. in a context where confounders are eliminated as far as possible. Note, however, that we do not bias our test in favor of pfPCH. In principle, eliminating the three confounders could work for or against the pfPCH-prediction.

2.3 Nash equilibria of the MG

Consider an MG with $n \geq 2$ and let $s$ be a pure-strategy profile with $c_k(s) \geq 2$ for all $k$ and $|M(s)| \geq 2$. By the tie-breaking rule, the smallest element of $M(s)$ is the minimizer and only the players having chosen this minimizer get a nonzero expected payoff. But no player has an incentive to deviate unilaterally, because switching to the minimizer causes this amount to lose its minimizer status and leads to a zero payoff. Hence $s$ is a Nash equilibrium. The MG has a variety of such asymmetric equilibria. However, all these asymmetric equilibria, requiring explicit coordination in a symmetric one-shot setting, are implausible solutions. Classical game theoretic choice
prediction would instead point to a symmetric Nash equilibrium of the MG. Such an equilibrium always exists, since the game is symmetric and finite.

The MG cannot be solved analytically and the simplifying approach of modeling the MG as a Poisson game (Myerson, 1998, 2000) does not work. However, for our purposes we do not need to explicitly calculate an equilibrium. Indeed, every symmetric equilibrium of the MG approaches the uniform distribution on the set $N$ of available choices as the number $I$ of players grows to infinity, and the same is true for the aggregate distribution of choices in any non-pure equilibrium. Since we have a large number of participants in our experiments, we can safely approximate the symmetric equilibrium by the uniform distribution. As an example, if $N = \{100, 150, 200\}$, as in our basic experimental treatment, then the equilibrium distribution is $E \approx (0.320, 0.329, 0.351)$ for $I = 300$ players. Whenever the subject pool is sufficiently large, the distribution of choice frequencies as predicted by the Nash hypothesis will be close to uniform.

### 2.4 PCH-predictions for the MG

What choice frequencies $(p_1, p_2, p_3)$ does the PCH model predict in the MG with $N = \{1, 2, 3\}$ in a large population? The answer depends on the exact values of $\tau$ and $I$, but for large $I$ the prediction is that $p_1 < p_2 < p_3$ holds uniformly for $0 < \tau < \bar{\tau}$, where $\bar{\tau} \approx 1.8$. To see this, note that the nonstrategic $L0$ types choose each number with probability $1/3$ by assumption. Hence $L1$ types believe that choices are distributed uniformly and maximize their

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6The latter route proved successful for LUPI games (Östling et al., 2011) and lowest unique bid auctions (Pigolotti et al., 2012), but the uniqueness of winning choices in these games, which the MG lacks, is crucial for the Poisson games approach to generate a tractable solution.

7Let $e$ be in the $\omega$-limit of the sets of symmetric equilibria for $I \to \infty$. Assume $e$ is not uniform. Then there are integers $\hat{k}$ and $\check{k}$ such that $e_{\hat{k}} > e_{\check{k}}$ are the maximal and minimal frequencies, respectively. But then by the law of large numbers we can make it arbitrarily more likely for $\check{k}$ to be the minimizer than $\hat{k}$ by choosing the number $I$ of players large enough. This contradicts equality of expected payoffs in mixed equilibria.
payoff by picking the highest number, 3. \( L2 \) types believe that all others are \( L0 \) or \( L1 \), hence in their opinion 3 will be chosen most often and 1 and 2 have equal chances of turning out as minimizers. \( L2 \) types therefore choose 2, if \( I \) is large. For the same reason \( L3 \) types opt for 1 as the minimizer. The choices of \( L4 \) and higher-level types are less straightforward, as they depend on the order of the frequencies of \( L1 \), \( L2 \), and \( L3 \). However, numerical computation shows that if \( 0 < \tau < \bar{\tau} \), then the lowest number, 1, has the lowest frequency according to the beliefs of \( L3 \) and all higher types, which therefore also pick 1, if \( I \) is not too small.

For \( \tau > \bar{\tau} \) the PCH-prediction is near and for \( \tau \to \infty \) converges to the equilibrium \( E \) which equals the uniform distribution for an infinite number of players. In the range \( \tau \in [0, \bar{\tau}] \), the PCH-predictions for increasing \( \tau \) describe a loop as depicted in Figure 1.\(^8\) The loop starts at \( E \), dives into the triangular section of the simplex where \( p_1 < p_2 < p_3 \), takes a turn at around \( \tau = 0.8 \) and heads back towards \( E \) until it intersects itself at \( \tau = \bar{\tau} \). The pfPCH point-prediction is \((p_1, p_2, p_3) \approx (0.266, 0.325, 0.409)\).

### 2.5 Level-k-predictions for the MG

Level-\( k \) models have several parameters, viz. the frequencies of the various types of players. These frequencies have to be estimated from the data or transferred from estimations in similar games, so unlike Nash equilibrium or the pfPCH model, an unconstrained level-\( k \) model does not give a point-prediction for the MG. For predictions we therefore take the generous approach to allow for all point-predictions possibly arising from the “typical” estimates outlined in section 1.2 above: The population predominantly consists of \( l_1 \) and \( l_2 \), complemented by smaller fractions of \( l_3 \), \( l_4 \), \( d_1 \), and equilibrium types. We call the level-\( k \) model with these characteristics the

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\(^{8}\)Technically, this loop is not smooth near \( E \) if \( I \) is finite, since for every given \( I \) there exists a threshold value of \( \tau \) below which the \( L2 \) type picks 3 instead of 2. But for large \( I \) this threshold value and the resulting discontinuity in the loop are so small that the latter is not visible in Figure 1.
standard level-k model or SLK model.

In the MG, the SLK model’s types $l_0$, $l_1$, and $l_2$ are behaviorally indistinguishable from the PCH model’s corresponding types $L_0$, $L_1$, and $L_2$. These types choose uniformly, 3, and 2, respectively. Note, however, that contrary to the PCH model higher types in the SLK model, by best responding to level-$(k - 1)$, never choose number 1 but switch between choosing 3 and 2 only. The dominance type $d_1$ behaves like $l_1$ and chooses 3, since there are no dominated strategies. Finally, the equilibrium type of the SLK model behaves like the $l_0$ type, since the symmetric equilibrium has an approximately uniform distribution of number choices. An important constraint in the SLK model is that, as $l_1$ and $l_2$ are predominant, any other type’s frequency is restricted to be at most $1/3$. For the MG this means that at least $2/3$ of the population chooses 3 or 2 (types $l_1$ to $l_4$ and $d_1$) and at most $1/3$ chooses uniformly (equilibrium type). This translates into the “prediction set” $\{(p_1, p_2, p_3) : p_1 \leq \min(p_2, p_3, 1/9)\}$ for the SLK model. This prediction set is depicted by the shaded area in Figure 1. Note that at most about 11% of all choices fall on the low amount 1, since this amount is only picked by $1/3$ of the equilibrium type, whose frequency is itself constrained to be less than $1/3$.

Apart from their different number of parameters, the major distinction between the PCH model and the SLK model is that in the SLK model players expect all other players to be of a single type and therefore to choose the same strategy, while in the PCH model players of type $L_2$ or higher expect others to be of different types and to choose potentially different strategies. In other words, players recognize heterogeneity in the population under PCH, but not under SLK.
3 Experiments

3.1 Basic setup and hypotheses

Since the predicted choice frequencies of the pfPCH model are not too far from the Nash equilibrium $E$, rejecting the Nash hypothesis with an adequate statistical power requires a large number of experimental subjects. We therefore conducted two large-scale online experiments. A total of 1360 subjects were recruited from first-year undergraduates at the Vienna University of Economics and Business. These subjects were unlikely to have been exposed to game theory, as this is only taught later on.

To provide appropriate, yet feasible incentives, we decided to use substantial amounts of money as prizes. The basic setup of the game required participants in the experiment to choose between the three amounts: €100, €150, and €200. All experiments presented in the following use variations of this basic game.

Based on the Nash equilibrium of this game for a sufficiently large subject
pool and on the prediction set of the SLK model, we can state the following two null-hypotheses:

\textbf{(H0a)} Choice frequencies are uniformly distributed ($p_1 = p_2 = p_3$).

\textbf{(H0b)} Choice frequencies belong to the SLK model’s prediction set \{($p_1, p_2, p_3$) : $p_1 \leq \min(p_2, p_3, 1/9)$\}.

Opposed to these two benchmarks, the predictions derived from the PCH model give rise to three increasingly sharp hypotheses:

\textbf{(H1)} The choice frequencies $p$ are ordered $p_1 < p_2 < p_3$.

\textbf{(H2)} The choice frequencies $p$ can be derived from a PCH model for some Poisson parameter $\tau$.

\textbf{(H3)} The choice frequencies $p$ can be derived from the pfPCH model, i.e. from the PCH model with Poisson parameter $\tau = 1.5$.

For hypotheses testing and reporting of significant results we apply a 5\% level of significance.

\textbf{3.2 Experiment 1}

When implementing a game for the first time, appropriate incentives are a main concern. Experimental studies frequently investigate whether behavior is sensitive to changes in incentives. This question becomes even more central in the domain of cognitive models, where incentives might crucially affect subjects’ cognitive effort. To address this issue appropriately, our first experiment varies the stakes of the game in a between-subjects design. We denote the basic setup of Experiment 1 described above as the \textit{low-stakes treatment} of Experiment 1. In addition, we introduced a \textit{high-stakes treatment} where the amounts to choose from were quadrupled to €400, €600 and €800. If choice distributions are equal across both treatments, the basic findings of our experiment are not the result of a specific level of incentives.
3.2.1 Procedure

To conduct the online experiment we chose to use the free software LimeSurvey. A cohort of 3824 first-year undergraduate students from the Vienna University of Economics and Business were invited by e-mail. The e-mail asked students to participate in an online experiment that could earn them a substantial amount of money while requiring only a few minutes of their time. To reduce the transaction costs of participating as much as possible, the invitation e-mail contained a link to the university web page hosting the experiment. Each invited student was randomly assigned to one of the two treatments and received a unique seven-digit identification number to ensure that he or she could participate only once. Additionally, the e-mail invitation told subjects how many fellow students had been invited to the experiment as well, which helped them to assess the strategic situation and create homogeneous expectations about the possible pool of participants. 1905 students were invited to the low stakes treatment, of which 312 (154 females and 158 males) actually took part. For the high stakes treatment, 1919 students were invited of which 305 (164 females and 141 males) actually participated. The experiment was open for participation for four days.

On the experiment website subjects were instructed that they will have to choose one of three options that correspond to three different payoff levels. They were told that the ‘winning number’ is the one that is chosen least often among all participants in the treatment. After their choice among the three amounts of money, subjects had to state their gender and age before submitting their choice and completing the experiment. The winning number was announced in an e-mail after the experiment was closed and all participants were invited for the public draw of the winner.

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9 After clicking the link to the web page in the invitation e-mail virtually no-one dropped out of the experiment.
10 For instructions and screen shots, see the Appendix.
3.3 Experiment 2

Experiment 2 replicates the low stakes treatment of Experiment 1, but here the three amounts to choose from were permuted to avoid possible framing by the increasing order in which the amounts were presented in Experiment 1.\textsuperscript{11} As in Experiment 1, a (fresh) cohort of first-year undergraduate students was approached by e-mail invitations. 3680 e-mails were sent, resulting in a total number of 743 participants (404 females and 339 males).\textsuperscript{12} All other procedural details were the same as in the previous experiment.

3.4 Results

As the predictions for the distribution of choices across the three options are independent of the stake size, we denote the three amounts – irrespective of the treatment – Small (corresponding to €100 or €400), Medium (€150 or €600) and Large (€200 or €800). Table 1 presents absolute (n) and relative (f) choice frequencies for the three amounts in each treatment.

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<th>Experiment</th>
<th>Experiment 1</th>
<th>Experiment 2</th>
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</tr>
<tr>
<td>Medium</td>
<td>108</td>
<td>0.35</td>
<td>103</td>
</tr>
<tr>
<td>Large</td>
<td>140</td>
<td>0.45</td>
<td>133</td>
</tr>
<tr>
<td>Total</td>
<td>312</td>
<td>1</td>
<td>305</td>
</tr>
</tbody>
</table>

\textsuperscript{11}Actually, Experiment 2 consisted of six rounds of the minimizer game. The intention of this was to enable the analysis of potential learning effects, the results of which will be reported elsewhere. Here we focus on one-shot games and therefore consider exclusively data from the first round of Experiment 2.

\textsuperscript{12}The slightly higher response rate to the invitations might be due to increased advertising efforts for experiments at the university.

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Table 1 suggests that the distribution of choices across the three possible amounts is very similar across treatments and experiments. The last two columns therefore show absolute and relative frequencies of choices when pooling all treatments. The minimizer is the small amount, while the most frequently chosen number is the large amount.

Table 2 presents the pooled choice frequencies vis-a-vis the Nash-prediction (Nash), the closest of all SLK-predictions (cSLK)\(^{13}\), and the parameter-free PCH-prediction (pfPCH). These point-predictions are also depicted in Figure 2. The overall choice distribution is significantly different from uniform \((\chi^2\text{-Test}, p < .0001)\), rejecting the Nash-prediction (hypothesis H0a). It is also significantly different from the closest of all SLK-predictions (hypothesis H0b, \(p < .0001\)). Moreover, the observed ranking \(p_1 < p_2 < p_3\) is highly significant in all pairwise comparisons (one sample test of proportions, \(p < .0001\) for each one). Hence we cannot reject hypothesis H1.\(^{14}\)

<table>
<thead>
<tr>
<th></th>
<th>Nash</th>
<th>cSLK</th>
<th>pfPCH</th>
<th>actual choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>0.33</td>
<td>0.11</td>
<td>0.27</td>
<td>0.23</td>
</tr>
<tr>
<td>Medium</td>
<td>0.33</td>
<td>0.40</td>
<td>0.33</td>
<td>0.35</td>
</tr>
<tr>
<td>Large</td>
<td>0.33</td>
<td>0.49</td>
<td>0.41</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Examining the PCH-loop closer reveals that \(\tau\)-values ranging from \(\tau = 1.26\) to \(\tau = 1.46\) predict choice frequencies which are statistically indistinguishable on a 5\% margin from the sample distribution based on our pooled data. Figure 3 illustrates this interval of \(\tau\)-values as the corresponding loop

\(^{13}\)This is the maximum-likelihood point-prediction within the prediction set of the SLK model, which is approximately \((0.11, 0.40, 0.49)\).

\(^{14}\)The background characteristics of participants allows to look into potential gender differences with respect to thinking steps. Tests on the equality of distributions across female and male participants, as well as across stakes for both females and males, do not reveal any significant differences.
segment of choice frequencies. Hence we cannot reject hypothesis H2. According to a maximum likelihood estimator, the best fitting value for $\tau$ under the assumption that the PCH model applies is $\tau = 1.37$ (with a corresponding $p$-value of 0.180).

The interval of $\tau$-values yielding PCH-predictions which are not significantly different from the data comes close to, but does not include the value $\tau = 1.5$ we used for our pfPCH model. Therefore, formally we have to reject hypothesis H3. This is not surprising, since our test is high-powered and the pfPCH model is unlikely to capture subjects’ beliefs and choices exactly. However, the pfPCH-prediction is remarkably close to the data, especially when compared with the rival models. From Figure 2 this seems obvious.

To quantify the accuracy of the three primary models’ predictions we show in Table 3 the normalized distance\textsuperscript{15} between predictions and experimental choice frequencies as well as the $p$-values of the $\chi^2$-tests associated to the

\textsuperscript{15}This is the Euclidean distance divided by 0.942, the maximum possible distance of a point in the 3-simplex from the choice frequencies.
Figure 3: PCH-predictions with $1.26 < \tau < 1.46$ are not significantly different from the pooled data.

Table 3: Prediction accuracy

<table>
<thead>
<tr>
<th></th>
<th>Nash</th>
<th>cSLK</th>
<th>pfPCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>norm. distance</td>
<td>$0.143$</td>
<td>$0.158$</td>
<td>$0.044$</td>
</tr>
<tr>
<td>$p$-value</td>
<td>$10^{-16}$</td>
<td>$10^{-44}$</td>
<td>$0.02$</td>
</tr>
</tbody>
</table>

The pfPCH-prediction clearly stands out compared to the Nash- and the cSLK-prediction. The runner-up, Nash equilibrium, has a normalized distance from the data which is larger by a factor of 3.25, and the probability of getting the experimental or more extreme data is larger by a factor of $2 \times 10^{14}$ for pfPCH than for Nash.
4 Other predictions

4.1 Alternative level-0 behavior

The usual assumption on level-0 behavior in the SLK- and the PCH model is that these players randomize uniformly over the set of pure strategies. This assumption can be applied to any finite game and is a natural choice for a general prediction model. There are, however, other suggestions as well. It might be argued that in the MG the nonstrategic approach would be to choose the highest amount. Indeed, among the five level-0 features discussed by Wright and Leyton-Brown (2014a), only maxmax payoff and minimax regret are selective in the MG, and both point to the large amount as the nonstrategic choice.

If we employ this alternative, the SLK model’s prediction-set does not change. However, the pfPCH-prediction does. The large amount is now chosen by $L0$ and $L3$ types, the medium amount is chosen by $L1$, while $L2$, $L4$ and all higher types choose the small amount. This results in an overall choice distribution which is very close to uniform, viz. the point-prediction $(0.32, 0.33, 0.35)$. This prediction has a normalized distance from the experimental choice distribution of 0.119 and can easily be rejected ($p < 10^{-11}$).

4.2 Asymmetric Nash equilibria

Thinking-steps models allow for heterogenous types, while symmetric Nash equilibrium does not. One might therefore want to include asymmetric equilibria into the Nash model. While we do not think that asymmetric Nash equilibria are meaningful as possible predictions of initial behavior in one-shot symmetric games with many players, we include them here for completeness.

The pure asymmetric Nash equilibrium profiles of the MG described in section 2.3 have the common structure that two amounts are chosen the same
number of times and the third amount is chosen more often. These equilibria lead to choice distributions which form three rays in the simplex, each connecting the center $E$ with one of the vertices. Among the points in this Nash-prediction-set the maximum likelihood prediction is $(0.29, 0.29, 0.43)$ with normalized distance of 0.086, which can also be rejected ($p < 10^{-6}$).

In the behavioral and psychological game theory literature there are other equilibrium concepts as well. While these do not explicitly try to capture initial choices, we include two of them in the following sections just to demonstrate that none of them can even substantially improve upon the Nash-prediction for the MG.

### 4.3 Quantal response equilibrium

Quantal response equilibrium (QRE) was introduced by McKelvey and Palfrey (1995). In a QRE, players adopt noisy best responses to each other’s choices. A common specification is logit equilibrium (LQRE), where players’ choices are based on the logistic quantal response function with parameter $\lambda$. A symmetric LQRE $p$ requires that $p_k = \frac{e^{\lambda u(k, p)}}{\sum_j e^{\lambda u(j, p)}}$, where $u(k, p)$ is the expected payoff of a player choosing amount $e_k$ while all others play the mixture $p$.

For the MG, the same argument as for Nash equilibrium (see footnote 7) shows that as the number of players goes to infinity, symmetric LQREs converge to the uniform distribution $E$ in the simplex for any value of $\lambda$. Hence for a large number of players the LQRE-prediction approximately coincides with NE and can therefore be rejected as well.

### 4.4 Impulse balance equilibrium

In a symmetric (generalized) impulse balance equilibrium (IBE), each player selects each pure strategy with a probability proportional to its expected impulse, the impulse of a pure strategy given an opponents’ pure strategy.
profile being the difference between this strategy’s payoff and the lowest payoff among all pure strategies (Chmura et al., 2014). In the MG the least payoff is zero for any opponents’ pure strategy profile and the impulse of a strategy is simply its payoff. In a mixed symmetric IBE $p$ the frequency $p_k$ is therefore its relative payoff $p_k = \frac{u(k,p)}{\sum_{j=1}^{3} u(j,p)}$. As for LQRE above, $p$ approximates the uniform distribution $E$ as the number of players grows large. The IBE-prediction can therefore be rejected.

5 Conclusion

We discussed the question why the PCH model (based on self-regarding preferences) seems to predict well in some games and fails in others. Based on conjectures about possible biases due to social preferences, complexity-induced infeasibility of maximizing behavior, and maximin-principle interference, we constructed a multi-player game, the minimizer game, that avoids these obstacles. We then formulated three increasingly sharp hypotheses from the PCH approach, where the last one corresponds to a context- and parameter-free prediction. We tested these hypotheses in two large-scale Internet experiments. Stake size did not appear to influence the distribution of thinking steps. Predictions derived from Nash equilibrium and the SLK model are clearly rejected by the data from our experiments, and also quantal response equilibrium and impulse balance equilibrium did not predict experimental choices well. We could not reject the hypothesis that the data result from some PCH model. While the hypothesis that the pfPCH model is the true model can be rejected, it nevertheless predicts remarkably well.

We did not believe that $\tau = 1.5$ is the best overall choice for the PCH model; we just used this suggestion to avoid any appearance of post hoc model fitting. But we think that our results show that when a game is “pure” and “simple”, i.e. stripped of all complications introduced by social preferences, algebraic complexities and risk issues, then the thinking steps approach to
predicting behavior is useful. Furthermore, under these circumstances the pfPCH model might be a better predictor than the SLK model. The reason for this seems to be that the latter ignores subjects’ taking into account that the behavior of others might be heterogenous. We therefore propose the pfPCH model as a useful context- and parameter-free alternative to Nash equilibrium in predicting initial choices in simple games.

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tainty, equilibrium selection, and coordination failure in average opinion


A Appendix: Instructions

These instructions have been translated from German. Original instructions are available from the authors upon request.

A.1 Experiment 1: Instructions and choice screen

Screen 1: Instructions

Thank you for participating in this online experiment!

Instructions

On the following page, you will find three amounts of money to choose from. Please select one of the three amounts!

The amount that is selected least often by all participants is the winning amount.

Of all participants who selected this winning amount, one will be randomly drawn as the winner. This participant will be paid out the winning amount.

Note:
To keep the chances of being drawn the winner, you have to select the amount that you believe will be selected least often in total.

Proceed >>

Screen 2: Choice in low (high) stakes treatment
A.2 Experiment 2: Instructions and choice screen

Screen 1: Instructions

Thank you for participating in this online experiment!

Instructions

1. This online experiment consists of 6 rounds. In each of the following 6 rounds, you must select one of 3 amounts of money.
2. When all participants have completed the experiment, one of the 6 rounds is randomly drawn. This round is called decision round.
3. The amount of money that is in the decision round selected least often by all participants is the winning amount.
4. Of all participants who have chosen the winning amount in the decision round, one participant is randomly drawn. She/he is notified by mail and receives the winning amount in cash.

Note: Each round can be the decision round. To keep the chances of being drawn the winner, you have to select the amount that you believe will be selected least often in this round.

Screen 2: Choice in round 1

Please choose one of the three amounts:

☐ 100 € (400 €)
☐ 150 € (600 €)
☐ 200 € (800 €)

Help: Here are the rules again: The amount that is chosen least often by all participants is the winning amount. Of all participants who have selected this winning amount, one will be randomly drawn as the winner. She/he receives this winning amount as payoff.

Proceed >>
Experiment 3

Screen 1: Instructions

Thank you for participating in this online experiment!

Instructions

1. This online experiment consists of 6 rounds. In each of the following 6 rounds, you must select one of 3 amounts of money.

2. When all participants have completed the experiment, one of the 6 rounds is randomly drawn. This round is called the decision round.

3. The amount of money that is in the decision round selected least often by all participants is the winning amount.

4. Of all participants who have chosen the winning amount in this round, one participant is randomly drawn. She/he is notified by mail and receives the winning amount in cash.

Note: Each round can be the decision round. To keep the chances of being drawn the winner, you have to select the amount that you believe will be selected least often in this round.

Screen 2: Choice

Round 1:
Please choose one of the three amounts:

☐ 100 €  ☐ 200 €  ☐ 150 €

Help: This is round 1. If this round is drawn as the decision round, the amount that is chosen least often by all participants in this round is the winning amount. Of all participants who have chosen the winning amount in this round, one will be randomly drawn as the winner. She/he receives the winning amount in cash.

Proceed >>

A.3 Both experiments: final screens

Screen: Gender

Please state your gender:

☐ Female
☐ Male

Proceed >>

Screen: Submission

To finally submit your choice(s), please click on the submit button.

Submit >>

Screen: Confirmation
Thank you for participating!

To confirm that we received your decision, you will obtain an e-mail shortly. In case you are the lucky winner, you will be notified by e-mail as well.