Necessary and sufficient conditions in the problem of optimal investment in incomplete markets

D. Kramkov*
Carnegie Mellon University

W. Schachermayer†
Vienna University of Technology

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Abstract

Following [10] we continue the study of the problem of expected utility maximization in incomplete markets. Our goal is to find *minimal* conditions on a model and a utility function for the validity of several key assertions of the theory to hold true. In [10] we proved that a minimal condition on the utility function *alone*, i.e. a minimal *market independent* condition, is that the asymptotic elasticity of the utility function is strictly less than 1. In this paper we show that a *necessary and sufficient* condition on *both*, the utility function and the model, is that the value function of the dual problem is finite.

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1 Introduction and Main Results

We study the same financial framework as in [10] and refer to this paper for more details and references. We consider a model of a security market which consists of $d+1$ assets, one bond and d stocks. We work in discounted terms, i.e., we suppose that the price of the bond is constant, and denote by $S = (S^i)_{1\le i\le d}$ the price process of the d stocks. The process S is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. Here T is a finite time horizon. To simplify notation we assume that $\mathcal{F} = \mathcal{F}_T$.

A (self-financing) portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio, and $H = (H^i)_{1\le i\le d}$ is a predictable S-integrable process, where H^i_t specifies, how many units of asset i are held in the portfolio at time t. The value process $X = (X_t)_{0\le t\le T}$ of such a portfolio Π is given by

$$X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \le t \le T. \tag{1}$$

We denote by $\mathcal{X}(x)$ the family of wealth processes with non-negative capital at any instant, i.e. $X_t \ge 0$ for all $t \in [0, T]$, and with initial value equal to x:

$$\mathcal{X}(x) = \{X \ge 0 : \ X \text{ is defined by (1) with } X_0 = x\}.$$

We shall use the shorter notation $\mathcal{X}$ for $\mathcal{X}(1)$. Clearly,

$$\mathcal{X}(x) = x\mathcal{X} = \{xX : \ X \in \mathcal{X}\}, \quad \text{for } x \ge 0.$$

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}$ is a local martingale under $\mathbb{Q}$. The family of equivalent local martingale measures will be denoted by $\mathcal{M}$. We assume throughout that

$$\mathcal{M} \neq \emptyset. \tag{2}$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [4], [5] for precise statements and references.

We also consider an economic agent in our model, whose preferences are modeled by a utility function $U : (0, \infty) \to \mathbf{R}$ for wealth at maturity time T. Hereafter we will assume that the function U is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$\begin{aligned} U'(0) &= \lim_{x\to 0} U'(x) = \infty, \\ U'(\infty) &= \lim_{x\to\infty} U'(x) = 0. \end{aligned} \tag{3}$$

For a given initial capital $x > 0$, the goal of the agent is *to maximize the expected value of terminal utility.* The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)]. \tag{4}$$

Intuitively speaking, the value function u plays the role of the utility function of the investor at time 0, if she subsequently invests in an optimal way. A well known tool in studying the optimization problem (4) is the use of duality relationships in the spaces of convex functions and semimartingales, see, for example, [1], [11], [8], [2], [3], [6], [7], [9], [10], [13].

The conjugate function V to the utility function U is defined as

$$V(y) = \sup_{x>0}[U(x) - xy], \quad y > 0. \tag{5}$$

It is well known (see, for example, [12]) that if U satisfies the hypotheses stated above, then V is a continuously differentiable, decreasing, strictly convex function satisfying $V'(0) = -\infty$ and $V'(\infty) = 0, V(0) = U(\infty), V(\infty) = U(0)$, and the following relation holds true

$$U(x) = \inf_{y>0}[V(y) + xy], \quad x > 0.$$

In addition the derivative of U is the inverse function of the negative of the derivative of V, i.e.

$$U'(x) = y \iff x = -V'(y).$$

Further we define the family $\mathcal{Y}$ of nonnegative semimartingales, which is dual to $\mathcal{X}$ in the following sense:

$$\mathcal{Y} = \{Y \geq 0 : \ Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}\}.$$

Note that, as $1 \in \mathcal{X}$, any $Y \in \mathcal{Y}$ is a supermartingale. Note also that the set $\mathcal{Y}$ contains the density processes of all $\mathbb{Q} \in \mathcal{M}$. For $y > 0$, we define

$$\mathcal{Y}(y) = y\mathcal{Y} = \{yY : \ Y \in \mathcal{Y}\}$$

and consider the following optimization problem:

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)]. \tag{6}$$

The next result from [10] shows that the value functions u and v to the optimization problems (4) and (6) are conjugate.

Theorem 1 ([10], Theorem 2.1) *Assume that (2) and (3) hold true and*

$$u(x) < \infty \quad \textit{for some} \quad x > 0. \tag{7}$$

Then:

1. *$u(x) < \infty$, for all $x > 0$, and there exists $y_0 \geq 0$ such that $v(y)$ is finitely valued for $y > y_0$. The value functions u and v are conjugate:*

$$\begin{aligned} v(y) &= \sup_{x>0}[u(x) - xy], \quad y > 0, \\ u(x) &= \inf_{y>0}[v(y) + xy], \quad x > 0. \end{aligned} \tag{8}$$

   The function u is continuously differentiable on $(0, \infty)$ and the function v is strictly convex on $\{v < \infty\}$.

   The functions u' and v' satisfy:

$$\begin{aligned} u'(0) &= \lim_{x \to 0} u'(x) = \infty, \\ v'(\infty) &= \lim_{y \to \infty} v'(y) = 0. \end{aligned}$$

2. *The optimal solution $\widehat{Y}(y) \in \mathcal{Y}(y)$ to (6) exists and is unique provided that $v(y) < \infty$.*

As in [10] we are interested in the following questions related to the optimization problems (4) and (6):

1. Does the optimal solution $\widehat{X} \in \mathcal{X}(x)$ to (4) exist?

2. Does the value function $u(x)$ satisfy the usual properties of a utility function, i.e., is it increasing, strictly concave, continuously differentiable and such that $u'(0) = \infty$, $u'(\infty) = 0$?

3. Does the dual value function v have the representation:

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \tag{9}$$

   where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}) = (\Omega, \mathcal{F}_T)$?

In [10] (see Theorem 2.2 and the counterexamples in Section 5) we proved that a minimal assumption on the utility function U, which implies positive answers to these questions for an *arbitrary* financial model, is the condition on the asymptotic behavior of the elasticity of U:

$$AE(U) \triangleq \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

The subsequent theorem, which is the main result of the present paper, and Note 1 below imply that a necessary and sufficient condition for all three assertions to have positive answers in the framework of a *particular* financial model is the finiteness of the dual value function.

Theorem 2 *Assume that (2) and (3) hold true and*

$$v(y) < \infty, \quad \forall y > 0. \tag{10}$$

Then in addition to the assertions of Theorem 1 we have:

1. *The value functions u and $-v$ are continuously differentiable, increasing and strictly concave on $(0,\infty)$ and satisfy:*

$$\begin{aligned} u'(\infty) &= \lim_{x\to\infty} u'(x) = 0, \\ -v'(0) &= \lim_{y\to 0} -v'(y) = \infty. \end{aligned}$$

2. *The optimal solution $\widehat{X}(x) \in \mathcal{X}(x)$ to (4) exists, for any $x > 0$, and is unique. In addition, if $y = u'(x)$ then*

$$U'\left(\widehat{X}_T(x)\right) = \widehat{Y}_T(y).$$

   where $\widehat{Y}(y) \in \mathcal{Y}(y)$ is the optimal solution to (6). Moreover, the process $\widehat{X}(x)\widehat{Y}(y)$ is a martingale.

3. *The dual value function v satisfies (9).*

Proof. Theorem 2 is a rather straightforward consequence of its "abstract version", Theorem 4 below. Admitting Theorem 4 as well as Proposition 1 below, the proof of Theorem 2 goes as follows.

For $x > 0$ and $y > 0$, let

$$\begin{aligned} \mathcal{C}(x) &= \left\{g \in \mathbf{L}^0(\Omega,\mathcal{F},\mathbb{P}) : \ 0 \le g \le X_T, \ \text{for some } X \in \mathcal{X}(x)\right\}, \quad (11) \\ \mathcal{D}(y) &= \left\{h \in \mathbf{L}^0(\Omega,\mathcal{F},\mathbb{P}) : \ 0 \le h \le Y_T, \ \text{for some } Y \in \mathcal{Y}(y)\right\}. \quad (12) \end{aligned}$$

In other words, $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are the sets of random variables dominated by the final values of elements from $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ respectively. With these notations the value functions u and v take the form:

$$\begin{aligned} u(x) &= \sup_{g\in\mathcal{C}(x)} \mathbb{E}[U(g)], \\ v(y) &= \inf_{h\in\mathcal{D}(y)} \mathbb{E}[V(h)]. \end{aligned}$$

According to Proposition 3.1 in [10] the sets $\mathcal{C}(x)$, $x > 0$, and $\mathcal{D}(y)$, $y > 0$, satisfy the conditions (16), (17) and (18) below. Hence Theorem 4 implies the assertions 1 and 2 of Theorem 2, except for the claim, that the product $\widehat{X}(x)\widehat{Y}(y)$ is a martingale. As regards this fact, note that $\widehat{X}(x)\widehat{Y}(y)$ is a positive supermartingale (by the construction of the set $\mathcal{Y}(y)$) and that we obtain the following equality from item 2 of Theorem 4:

$$\mathbb{E}[\widehat{X}_T(x)\widehat{Y}_T(y)] = xy = \widehat{X}_0(x)\widehat{Y}_0(y).$$

This readily implies the martingale property of $\widehat{X}(x)\widehat{Y}(y)$.

To prove the final assertion 3, we use Proposition 1 below. We denote by $\widetilde{\mathcal{D}}$ the set of Radon-Nikodym derivatives of equivalent martingale measures:

$$\widetilde{\mathcal{D}} = \left\{ h = \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad \mathbb{Q} \in \mathcal{M} \right\}.$$

The set $\widetilde{\mathcal{D}}$ is closed under countable convex combinations. In addition,

$$g \in \mathcal{C} \Leftrightarrow g \geq 0 \ \text{ and } \ \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \quad \forall \mathbb{Q} \in \mathcal{M}$$

by the general duality relationships between the terminal values of strategies and the densities of equivalent martingale measures (see [4] and [5]). Hence the set $\widetilde{\mathcal{D}}$ satisfies the assumptions of Proposition 1 and the result follows. □

Note 1 In view of the duality relation (8), condition (10) is equivalent to

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0,$$

which may equivalently be restated as

$$\lim_{x\to\infty} \frac{u(x)}{x} = 0.$$

In particular, this shows the necessity of (10) for Theorem 2 to hold true.

Note 2 In [10] (Theorem 2.2) we proved that the assertions of Theorem 2 follow from the assumptions of Theorem 1 and the condition $AE(U) < 1$ on the asymptotic elasticity of U. Let us now deduce this result as an easy consequence of Theorem 2.

We need to show that $AE(U) < 1$ implies that $v(y) < \infty$ for all $y > 0$. By Theorem 1 there is $y_0 > 0$ such that

$$v(y) < \infty, \quad y > y_0. \tag{13}$$

Further, the condition $AE(U) < 1$ is equivalent to the following property of V (see Lemma 6.3 in [10]): there are positive constants c_1 and c_2 such that

$$V\left(\frac{y}{2}\right) \leq c_1 V(y) + c_2, \quad y > 0. \tag{14}$$

The finiteness of v now follows from (13) and (14).

Note 3 Condition (10) may also be stated in the following equivalent form:

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty, \quad \forall y > 0. \tag{15}$$

Indeed, the implication (15) $\Rightarrow$ (10) is trivial, as the density processes of martingale measures belong to $\mathcal{Y}$. The more difficult reverse implication follows from Theorem 2.

2 The Abstract Version of the Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be non-empty sets of positive random variables such that

1. the set $\mathcal{C}$ is bounded in $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ and contains the constant function $g = 1$:

$$\lim_{n\to\infty} \sup_{g\in\mathcal{C}} \mathbb{P}[|g| \geq n] = 0 \tag{16}$$

$$1 \in \mathcal{C} \tag{17}$$

2. the sets $\mathcal{C}$ and $\mathcal{D}$ satisfy the bipolar relations:

$$\begin{aligned} g \in \mathcal{C} \quad &\Leftrightarrow \quad g \geq 0 \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall h \in \mathcal{D} \\ h \in \mathcal{D} \quad &\Leftrightarrow \quad h \geq 0 \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall g \in \mathcal{C} \end{aligned} \tag{18}$$

For $x > 0$ and $y > 0$, we define the sets

$$\begin{aligned} \mathcal{C}(x) &= x\mathcal{C} = \{xg : \ g \in \mathcal{C}\}, \\ \mathcal{D}(y) &= y\mathcal{D} = \{yh : \ h \in \mathcal{D}\}, \end{aligned}$$

and the optimization problems:

$$u(x) = \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)], \tag{19}$$

$$v(y) = \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)]. \tag{20}$$

Here $U = U(x)$ and $V = V(y)$ are the functions defined in Section 1. If $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are defined by (11) and (12), these value functions coincide with the value functions defined in (4) and (6).

Let us recall the following result from [10], which is the abstract version of Theorem 1.

Theorem 3 (Theorem 3.1 in [10]) *Assume that the sets $\mathcal{C}$ and $\mathcal{D}$ satisfy (16), (17) and (18). Assume also that the utility function U satisfies (3) and that*

$$u(x) < \infty \text{ for some } x > 0. \tag{21}$$

Then

1. $u(x) < \infty$, *for all* $x > 0$, *and there exists* $y_0 \geq 0$ *such that* $v(y)$ *is finitely valued for* $y > y_0$. *The value functions* u *and* v *are conjugate:*

$$v(y) = \sup_{x>0}[u(x) - xy], \quad y > 0, \tag{22}$$

$$u(x) = \inf_{y>0}[v(y) + xy], \quad x > 0.$$

   The function u *is continuously differentiable on* $(0, \infty)$, *and the function* v *is strictly convex on* $\{v < \infty\}$.

   The functions u' *and* $-v'$ *satisfy:*

$$u'(0) = \lim_{x \to 0} u'(x) = \infty,$$

$$v'(\infty) = \lim_{y \to \infty} v'(y) = 0.$$

2. *If* $v(y) < \infty$, *then the optimal solution* $\widehat{h}(y) \in \mathcal{D}(y)$ *to (19) exists and is unique.*

We now state the abstract version of Theorem 2. This theorem refines Theorem 3.2 in [10] in the sense that the condition $AE(U) < 1$ is replaced by the weaker condition (23) requiring the finiteness of the function $v(y)$, for all $y > 0$.

Theorem 4 *Assume that the utility function U satisfies (3), the sets $\mathcal{C}$ and $\mathcal{D}$ satisfy (16), (17) and (18), and that the value function v defined in (20) is finite:*

$$v(y) < \infty, \quad \forall y > 0. \tag{23}$$

Then, in addition to the assertions of Theorem 3, we have:

1. *The value functions u and $-v$ are continuously differentiable, increasing and strictly concave on $(0, \infty)$ and satisfy:*

$$u'(\infty) = \lim_{x \to \infty} u'(x) = 0,$$
$$-v'(0) = \lim_{y \to 0} -v'(y) = \infty.$$

2. *The optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ to (19) exists, for all $x > 0$, and is unique. In addition, if $y = u'(x)$, then*

$$\begin{aligned} U'(\widehat{g}(x)) &= \widehat{h}(y), \\ \text{and} \quad \mathbb{E}[\widehat{g}(x)\widehat{h}(y)] &= xy, \end{aligned}$$

where $\widehat{h}(y) \in \mathcal{D}(y)$ is the optimal solution to (20).

The proof of Theorem 4 is based on the following lemma.

Lemma 1 *Assume that the set $\mathcal{C}$ satisfies (16), (17) and (18) and the value function $u(x)$ defined in (19) is finite (for some or, equivalently, for all $x > 0$) and satisfies*

$$\lim_{x \to \infty} \frac{u(x)}{x} = 0. \tag{24}$$

Then the optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ exists for all $x > 0$.

Proof. The assertion that $u(x) < \infty$, for some $x > 0$, iff $u(x) < \infty$, for all $x > 0$, is a straightforward consequence of the concavity and monotonicity of u and the fact that $u \geq U$. Also observe that, as remarked in Note 1, assertion (24) is equivalent to (23).

Fix $x > 0$. Let $(f^n)_{n \geq 1}$ be a sequence in $\mathcal{C}(x)$ such that

$$\lim_{n \to \infty} \mathbb{E}[U(f^n)] = u(x).$$

We can find a sequence of convex combinations $g^n \in \mathrm{conv}(f^n, f^{n+1}, \ldots)$ which converges almost surely to a random variable $\widehat{g}$ with values in $[0, \infty]$, see, for example, [4], Lemma A1.1. Since the set $\mathcal{C}(x)$ is bounded in $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ we deduce that $\widehat{g}$ is almost surely finitely valued. By (18) and Fatou's lemma, $\widehat{g}$ belongs to $\mathcal{C}(x)$. We claim that $\widehat{g}$ is the optimal solution to (19), i.e.

$$\mathbb{E}[U(\widehat{g})] = u(x).$$

Let us denote by U^+ and U^- the positive and negative parts of the function U. From the concavity of U we deduce that

$$\lim_{n\to\infty} \mathbb{E}[U(g^n)] = u(x)$$

and from Fatou's lemma that

$$\liminf_{n\to\infty} \mathbb{E}[U^-(g^n)] \geq \mathbb{E}[U^-(\widehat{g})].$$

The optimality of $\widehat{g}$ will follow if we show that

$$\lim_{n\to\infty} \mathbb{E}[U^+(g^n)] = \mathbb{E}[U^+(\widehat{g})]. \tag{25}$$

If $U(\infty) \leq 0$, then there is nothing to prove. So we assume that $U(\infty) > 0$.

The validity of (25) is equivalent to the uniform integrability of the sequence $(U^+(g^n))_{n\geq 1}$. If this sequence is not uniformly integrable then, passing if necessary to a subsequence still denoted by $(g^n)_{n\geq 1}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A^n)_{n\geq 1}$ of $(\Omega, \mathcal{F})$, i.e.

$$A^n \in \mathcal{F}, \quad A^i \cap A^j = \emptyset, \quad \text{if } i \neq j,$$

such that

$$\mathbb{E}\left[U^+(g^n) I(A^n)\right] \geq \alpha, \quad \text{for } n \geq 1.$$

We define the sequence of random variables $(h^n)_{n\geq 1}$:

$$h^n = x_0 + \sum_{k=1}^{n} g^k I(A^k),$$

where

$$x_0 = \inf\{x > 0 : \ U(x) \geq 0\}.$$

For any $f \in \mathcal{D}$

$$\mathbb{E}[h^n f] \leq x_0 + \sum_{k=1}^{n} \mathbb{E}[g^k f] \leq x_0 + nx.$$

Hence $h^n \in \mathcal{C}(x_0 + nx)$. On the other hand

$$\mathbb{E}[U(h^n)] \geq \sum_{k=1}^{n} \mathbb{E}\left[U^+(g^k)I(A^k)\right] \geq \alpha n,$$

and therefore

$$\limsup_{x\to\infty} \frac{u(x)}{x} \geq \limsup_{n\to\infty} \frac{\mathbb{E}[U(h^n)]}{x_0 + nx} \geq \limsup_{n\to\infty} \frac{\alpha n}{x_0 + nx} = \alpha > 0.$$

This contradicts (24). Therefore (25) holds true. □

Proof of Theorem 4. Since, for $x > 0$ and $y > 0$,

$$U(x) \leq V(y) + xy,$$

and, for $g \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$,

$$\mathbb{E}[gh] \leq xy,$$

we have

$$u(x) \leq v(y) + xy.$$

In particular, the finiteness of $v(y)$, for some $y > 0$, implies the finiteness of $u(x)$, for all $x > 0$. It follows that the conditions of Theorem 3 hold true.

From the assumption that $v(y) < \infty$, $y > 0$, and the duality relations (22) between u and v, we deduce that

$$\lim_{x\to\infty} \frac{u(x)}{x} = \lim_{x\to\infty} u'(x) = 0. \tag{26}$$

Lemma 1 now implies that the optimal solution $\widehat{g}(x)$ to (19) exists, for any $x > 0$. The strict concavity of U implies the uniqueness of $\widehat{g}(x)$ as well as the fact that the function u is strictly concave too. The remaining assertions of item 1 related to the function v follow from the established properties of u, because of the duality relations (22) (see, for example, [12]).

Let $x > 0$, $y = u'(x)$, $\widehat{g}(x)$ and $\widehat{h}(y)$ be the optimal solutions to (19) and (20) respectively. We have

$$\begin{aligned} \mathbb{E}\left[\left|V(\widehat{h}(y) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))\right|\right] &= \\ \mathbb{E}\left[V(\widehat{h}(y)) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))\right] &\leq \\ v(y) + xy - u(x) &= 0, \end{aligned}$$

where, in the last step, we have used the relation $y = u'(x)$. It follows that

$$U(\widehat{g}(x)) = V(\widehat{h}(y)) + \widehat{g}(x)\widehat{h}(y).$$

This readily implies that

$$U'(\widehat{g}(x)) = \widehat{h}(y), \quad \text{a.s.}$$

and

$$\mathbb{E}[\widehat{g}(x)\widehat{h}(y)] = \mathbb{E}[U(\widehat{g}(x))] - \mathbb{E}[V(\widehat{h}(y))] = u(x) - v(y) = xy.$$

□

We complete the section with Proposition 1, which was used in the proof of item 3 of Theorem 2. This proposition was proved in [10] under the additional assumption $AE(U) < 1$.

Let $\widetilde{\mathcal{D}}$ be a convex subset of $\mathcal{D}$ such that

1. For any $g \in \mathcal{C}$
$$\sup_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[gh] = \sup_{h \in \mathcal{D}} \mathbb{E}[gh]. \tag{27}$$

2. The set $\widetilde{\mathcal{D}}$ is closed under countable convex combinations, i.e., for any sequence $(h^n)_{n\geq 1}$ of elements of $\widetilde{\mathcal{D}}$ and any sequence of positive numbers $(a^n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} a^n = 1$ the random variable $\sum_{n=1}^{\infty} a^n h^n$ belongs to $\widetilde{\mathcal{D}}$.

Proposition 1 *Assume that the conditions of Theorem 4 hold true and that $\widetilde{\mathcal{D}}$ satisfies the above assertions. The value function $v(y)$ defined in (20) then satisfies*

$$v(y) = \inf_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[V(yh)]. \tag{28}$$

The proof of the proposition will use the following two lemmas.

The first is an easy result, whose proof is analogous to the proof of Proposition 3.1 in [10] and therefore skipped.

Lemma 2 *Under the assumptions of Proposition 1, let $\widehat{h}(y)$ be the optimal solution to (20). Then there exists a sequence $(h^n)_{n\geq 1}$ in $\widetilde{\mathcal{D}}$, that converges almost surely to $\widehat{h}(y)/y$.* □

Lemma 3 *Under the assumptions of Proposition 1, we have, for each $y > 0$,*

$$\inf_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[V(yh)] < \infty.$$

Proof. To simplify the notation we shall prove the assertion of the lemma for the case $y = 1$.

Let $(\lambda_n)_{n\geq 1}$ be a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$. We denote by $\widehat{h}(\lambda_n)$ the optimal solution to (20) corresponding to the case $y = \lambda_n$. Let $(\delta_n)_{n\geq 2}$ be a sequence of strictly positive numbers, decreasing to 0, such that

$$\sum_{n=1}^{\infty} \mathbb{E}\left[V(\widehat{h}(\lambda_n))I(A_n)\right] < \infty, \text{ if } A_n \in \mathcal{F}, \mathbb{P}[A_n] \leq \delta_n, \quad n \geq 2. \tag{29}$$

From Lemma 2 we deduce the existence of a sequence $(h_n)_{n\geq 1}$ in $\widetilde{D}$ such that

$$\mathbb{P}\left[V(\lambda_n h_n) > V(\widehat{h}(\lambda_n)) + 1\right] \leq \delta_{n+1}, \quad n \geq 1.$$

We define the sequence of measurable sets $(A_n)_{n\geq 1}$ as follows:

$$\begin{aligned} A_1 &= \{V(\lambda_1 h_1) \leq V(\widehat{h}(\lambda_1) + 1\} \\ &\vdots \\ A_n &= \{V(\lambda_n h_n) \leq V(\widehat{h}(\lambda_n) + 1\} \setminus \bigcup_{k=1}^{n-1} A_k. \end{aligned}$$

This sequence has the following properties:

$$\begin{aligned} A_i \cap A_j &= \emptyset \text{ if } i \neq j, \\ \mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] &= 1 \\ \mathbb{P}[A_n] &\leq \delta_n, \quad n \geq 2. \end{aligned}$$

We define

$$h = \sum_{n=1}^{\infty} \lambda_n h_n$$

We have $h \in \widetilde{\mathcal{D}}$, because the set $\widetilde{D}$ is closed under countable convex combinations. The proof now follows from the inequalities:

$$\begin{aligned} \mathbb{E}[V(h)] = \sum_{n=1}^{\infty} \mathbb{E}[V(h)I(A_n)] &\overset{(i)}{\leq} \sum_{n=1}^{\infty} \mathbb{E}[V(\lambda_n h_n)I(A_n)] \\ &\overset{(ii)}{\leq} \sum_{n=1}^{\infty} \mathbb{E}\left[V(\widehat{h}(\lambda_n))I(A_n)\right] + 1 \overset{(iii)}{<} \infty, \end{aligned}$$

where (i) holds true because V is a decreasing function, (ii) follows from the construction of the sequence $(A_n)_{n\geq 1}$, and (iii) is a consequence of (29). $\square$

Proof of Proposition 1. Fix $\epsilon > 0$ and $y > 0$. We have to show that there is $h \in \widetilde{D}$ such that

$$\mathbb{E}\left[V((y+\epsilon)h)\right] \leq v(y) + \epsilon.$$

Let $\widehat{h} = \widehat{h}(y)$ be the optimal solution to the optimization problem (20) and f be an element of $\widetilde{\mathcal{D}}$ such that

$$\mathbb{E}[V(\epsilon f)] < \infty.$$

The existence of such a function f follows from Lemma 3. Let $\delta > 0$ be a sufficiently small number such that:

$$\mathbb{E}\left[(|V(\widehat{h})| + |V(\epsilon f)|)I(A)\right] \leq \frac{\epsilon}{2}, \quad \text{if } A \in \mathcal{F},\ \mathbb{P}[A] \leq \delta. \tag{30}$$

From Lemma 2 we deduce the existence of $g \in \widetilde{\mathcal{D}}$ such that

$$\mathbb{P}\left[V(yg) > V(\widehat{h}) + \frac{\epsilon}{2}\right] \leq \delta. \tag{31}$$

Denote

$$A = \left\{V(yg) > V(\widehat{h}) + \frac{\epsilon}{2}\right\},$$

and define

$$h = \frac{yg + \epsilon f}{y + \epsilon}.$$

Since the set $\widetilde{\mathcal{D}}$ is convex, $h \in \widetilde{\mathcal{D}}$. The proof now follows from the inequalities:

$$\mathbb{E}[V((y+\epsilon)h)] = \mathbb{E}[V(yg + \epsilon f)] \overset{(i)}{\leq} \mathbb{E}[V(yg)I(A^c)] + \mathbb{E}[V(\epsilon f)I(A)] \overset{(ii)}{\leq} v(y) + \epsilon$$

where (i) holds true, because V is a decreasing function, and (ii) follows from (30) and (31). $\square$

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