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Abstract

We revisit the influential economic growth model by Lucas (1988) ["On the mechanics of economic development." *Journal of Monetary Economics*, 22(1):3-42], assuming that households optimally allocate consumption and education over the life-cycle given an exogenous interest rate and exogenous wages. We show that in such a partial equilibrium setting, the original two-state (physical capital and human capital) optimization problem can be decomposed into two single-state optimal control models. This transformation allows us to rigorously prove the existence of a singular control describing the allocation of education time along a balanced growth path. We derive a constructive condition for a singular control to exist and show that under this condition infinitely many singular controls are optimal in the individual household problem. In contrast to the original general equilibrium framework in which an agent always chooses part-time education and part-time work, in our framework such an agent might find it optimal to allocate her whole available time to education at the beginning of her life and to focus on labor supply only when she is older.

Keywords: Optimal lifetime education, optimal control, singular control, economic growth, human capital.

JEL Classification: C60, O41, I20.

1 Introduction

In his seminal paper "On the Mechanics of Economic Growth" Robert E. Lucas presents an economic growth model where homogeneous individuals maximize their lifetime utility by choosing an optimal consumption path as well as the optimal allocation of their available time in education (human capital accumulation) and production. The problem that the individual faces in the Lucas model is specified as an optimal control problem with two state variables: physical capital and human capital. The existence of a unique solution along a balanced growth path is derived under the assumptions of (i) perfect foresight, (ii) perfect competition in labor and capital markets, and (iii) that the aggregate decisions of agents have repercussions on interest rates and wages through general equilibrium mechanisms.

In this paper we provide a rigorous analytical assessment of the Lucas (1988) model in a partial equilibrium reformulation. Such an analysis provides new insights into the mechanics of the model and delivers new results that can be useful when analyzing the mechanics of individual education decisions for a given interest rate and a given wage rate. In particular, we show that the partial equilibrium version of the Lucas model can be rewritten as a two-stage optimization problem in which the choice with respect to time devoted to work versus time devoted to human capital accumulation and the choice with respect to consumption can be disentangled from each other. In the first stage, agents optimally choose the division of time between human capital accumulation and production so as to maximize lifetime income, while in the second stage agents distribute consumption optimally over their infinite lives, given their lifetime income.

As our central result, we show that infinitely many optimal solutions exist for the time allocation between work and education in the partial equilibrium formulation of the Lucas model. For example, two feasible optimal strategies for individuals could be (i) to allocate the whole available time to education when young and to focus on labor supply when old (ii) to switch back and forth between full-time education and full-time work. The reason for such a result is that there are no repercussions of individual labor market decisions on the equilibrium wage rate. We show that the general equilibrium setting in Lucas (1988) acts as one equilibrium selection rule (of potentially many). However, other mechanisms such as nonlinearities in human capital accumulation, can also lead to the selection of a unique singular control.

The paper is organized as follows. Section 2 briefly presents the standard Lucas model of economic growth. Section 3 introduces the partial equilibrium reformulation of the model, together with the transformations of the model based on a two-stage formulation. Section 4 solves the partial equilibrium Lucas model. In Section 5, we present two possible equilibrium selection mechanisms (general equilibrium and nonlinearities in human capital accumulation dynamics). Section 6 concludes and presents paths of further research.

2 The Lucas Model of Economic Growth

The model of Lucas (1988) builds on the optimization problem of a representative agent who maximizes the discounted flow of utility derived from consumption over her infinite life. At every point in time, t, the agent is endowed with one unit of time that can be split between the part of it which is dedicated to the production of output, $u(t)$, and to the production of human capital, $1-u(t)$. Effective labor (i.e., human capital augmented labor input), $u(t)h(t)$, is compensated by the wage rate, denoted as $w(t)$. For savings, the interest rate $r(t)$ is paid. Instantaneous utility is iso-elastic and future utility is discounted with the time preference rate $\rho > 0$. The problem of the representative individual is thus given by

$$\max_{c(\cdot),u(\cdot)} \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} \, \mathrm{d}t, \tag{1}$$

subject to

$$\dot{k}(t) = r(t)k(t) + w(t)h(t)u(t) - c(t), \qquad k(0) = k_0 > 0, \tag{2}$$

$$\dot{h}(t) = \chi h(t)[1 - u(t)], \qquad h(0) = h_0 > 0, \tag{3}$$

$$u(t) \in [0,1], \qquad c(t) \geq 0, \qquad \lim_{t\to\infty} e^{-\int_0^t r(s)\,\mathrm{d}s} k(t) \geq 0, \tag{4}$$

where $c(t)$ denotes consumption per capita at time t, $k(t)$ refers to physical capital per capita, $h(t)$ denotes human capital per capita, and $\theta > 0$ and $\chi > 0$ are parameters measuring the relative risk aversion of individuals and the productivity of education, respectively. The *No Ponzi Game Restriction*, $\lim_{t\to\infty} e^{-\int_0^t r(s)\,\mathrm{d}s} k(t) \geq 0$, rules out infinite borrowing to finance consumption. Although it is not imposed explicitly by Lucas (1988), it follows implicitly from the transversality condition of the optimization problem.

Assuming $N(t)$ homogeneous agents, aggregate physical capital is given by $K(t) = N(t)k(t)$ and aggregate effective labor in the production process amounts to $L(t) = N(t)h(t)u(t)$. Both aggregate physical capital and aggregate effective labor are used to produce output $Y(t)$ according to a Cobb-Douglas production function with constant technology $A > 0$ and an elasticity of output with respect to physical capital of $0 < \alpha < 1$,

$$Y(t) = A\, K(t)^\alpha L(t)^{1-\alpha}.$$

Output can be consumed or invested in physical capital. The wage rate per unit of effective labor $w(t)$ and the capital rental rate $R(t)$ are determined on competitive markets and thus profit maximization at the firm level (assuming that the price of output is chosen as the numeraire) implies

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = A(1-\alpha)\left[\frac{K(t)}{L(t)}\right]^\alpha,$$

$$R(t) = \frac{\partial Y(t)}{\partial K(t)} = A\alpha\left[\frac{L(t)}{K(t)}\right]^{1-\alpha}.$$

The (net) interest rate paid to capital owners is given by $r(t) = R(t) - \delta$, where $\delta > 0$ is the depreciation rate of capital. Without loss of generality, we assume a stationary population and normalize $N(t) = 1$ for all t.

As compared to Lucas (1988), the model put forward is simpler because it abstracts from population growth and from externalities of human capital. For the optimization problem, Lucas (1988) derives a balanced growth path in the general equilibrium setting. The optimal equilibrium allocation for $u(t)$ is thus determined as the market equilibrium between firms and households. Lucas (1988) finds a unique and singular solution ($0 < u^*(t) < 1$ for all t). In our contribution, we aim to gain more insight into the singular solution of the optimal control model by analyzing the problem of the representative agent in a partial equilibrium setting and assessing potential solution selection mechanisms in this setting.

3 Reformulating the Lucas Model in a Partial Equilibrium Setting

In the partial equilibrium reformulation of the model we assume that the paths of the interest rate and the wage rate, $r(t)$ and $w(t)$, are given, known to the agent and not affected by the agent's decisions on savings and/or labor market participation. The representative agent maximizes thus her discounted lifetime utility given by equation (1), subject to constraints (2)–(4), conditional on given paths of $r(t)$ and $w(t)$. Assume in addition that $w(\cdot)$ is a continuously differentiable function.

The optimization problem given by (1)–(4) can be reformulated as a two-stage problem. The agent first chooses optimally the division of time between production and education in order to maximize her total discounted income over the life cycle. In the second stage, the agent decides on the optimal distribution of consumption over the life cycle, given income. In the model, the feasibility of such a decomposition rests on the assumptions of perfect capital markets and of the lack of disutility of work beyond the opportunity costs of investing in human capital. At any given point in time, agents have unconstrained access to the credit market at the prevailing interest rate, without frictions or credit constraints. Such a decomposition is not feasible if imperfections in the credit market are present. For instance, if there exists a credit constraint that restricts the amount of capital the agent can borrow, the specific path of income may matter for the distribution of consumption over time. At the beginning of the life cycle the constraint might be binding. In this case, the agent is forced to work if she wants to consume. This restricts the agent in her decisions on optimal education and consumption and renders the decomposition of the original Lucas model into a two stage optimization problem infeasible. The decomposition is also not possible in case that leisure is considered as an additional source of utility in a nonlinear way.

The equivalence result requires the assumption that the positive-valued function $w(t)$ is such that for any admissible control $u(t)$ and the corresponding trajectory $h(t)$,

$$\int_0^\infty e^{-r(t,0)} w(t)h(t)u(t)\,\mathrm{d}t < \infty,$$

where $r(t,s) = \int_s^t r(\tau)\,\mathrm{d}\tau$ is the accumulated interest rate between time s and time t (Assumption A0). If there was some control $u(t)$ for which the integral above was infinite, the No Ponzi Game constraint (4) would imply infinite consumption at time $t = 0$, which would be economically infeasible. The finiteness of total discounted lifetime income is ensured by adopting this assumption.

Proposition 1. *Let Assumption A0 hold. Then the original problem (1)–(4) is equivalent to the combination of the problem of choosing the division of time between production and education in order to maximize total discounted income and the problem of choosing a consumption path given maximum total discounted income.*

Consider an arbitrary admissible path $u(t)$ and its corresponding human capital path $h(t)$ being a solution to (3). Let $x(t)$ and $z(t)$ be solutions to

$$\dot{x}(t) = r(t)x(t) - c(t), \qquad x(0) = k_0 + \int_0^\infty e^{-r(t,0)} w(t)h(t)u(t)dt \tag{5}$$

and

$$\dot{z}(t) = r(t)z(t) + w(t)h(t)u(t), \qquad z(0) = -\int_0^\infty e^{-r(t,0)} w(t)h(t)u(t)dt. \tag{6}$$

Then, $k(t) = x(t) + z(t)$ for all $t \geq 0$. Note from (6) that

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \int_t^\infty e^{-r(s,0)} w(t)h(t)u(t)dt = 0$$

and the No Ponzi Game condition (4) leads to

$$\lim_{t\to\infty} e^{-r(t,0)} k(t) = \lim_{t\to\infty} e^{-r(t,0)} x(t) + \lim_{t\to\infty} e^{-r(t,0)} z(t) = \lim_{t\to\infty} e^{-r(t,0)} x(t) \geq 0. \tag{7}$$

The state variable $z(t)$ therefore does not play any role for the utility optimization problem (1) and can be omitted in further considerations.

Consider the No Ponzi Game condition (7). From (5) it follows that

$$x(t) = e^{r(t,0)} \left[k_0 + \int_0^\infty e^{-r(s,0)} w(s)h(s)u(s)\,\mathrm{d}s - \int_0^t e^{-r(s,0)} c(s)\,\mathrm{d}s \right]$$

and so (7) requires that

$$\int_0^\infty e^{-r(s,0)} c(s)\,\mathrm{d}s \leq k_0 + \int_0^\infty e^{-r(s,0)} w(s)h(s)u(s)\,\mathrm{d}s.$$

This inequality implies that the set of admissible controls $c(t)$ is the largest if $\int_0^\infty e^{-r(s,0)} w(s)h(s)u(s)\,\mathrm{d}s$ takes the maximal possible value. Thus, maximizing the right hand side will allow maximizing utility as given by (1). Therefore, the original problem (1)–(4) is reducible to the two separate problems put forward above.

We define *the education problem* to be the maximization problem of the total discounted lifetime income given by

$$\max_{u(t)} \int_0^\infty e^{-r(t,0)} w(t)u(t)h(t)\,\mathrm{d}t, \tag{8}$$

subject to

$$\dot{h}(t) = \chi h(t)[1 - u(t)], \qquad h(0) = h_0. \tag{9}$$

Note that the dynamic decision of optimal consumption allocation is taken with a static initial condition that corresponds to the constraint that is set by the total discounted lifetime income. Thus the education problem is defined as follows

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1 - \theta}\,\mathrm{d}t, \tag{10}$$

subject to

$$\dot{x}(t) = r(t)x(t) - c(t), \qquad x(0) = x_0 + \int_0^\infty e^{-r(t,0)} w(t)u(t)h(t)\,\mathrm{d}t, \tag{11}$$

$$\lim_{t\to\infty} e^{-r(t,0)} x(t) \geq 0. \tag{12}$$

Problem (10)-(12) is referred to as *consumption problem*.

The *consumption problem* has been studied in the literature and its optimal solution is given by the following expressions

$$c^*(t) = c_0^* e^{\frac{r(t,0)-\rho t}{\theta}}, \qquad c_0^* = \frac{x(0)}{\int_0^\infty e^{-r(t,0)} e^{\frac{r(t,0)-\rho t}{\theta}}\,\mathrm{d}t}. \tag{13}$$

The proof can be found in standard textbooks, (see, e.g., Acemoglu, 2009, pp. 294-298).

This decomposition simplifies the mathematical analysis of the original model considerably by reducing a two-state control optimization problem to two optimization problems each having one state variable and one control. Intuitively, this result proves that we do not need to think of agents as making their decisions on $u(t)$ and $c(t)$ simultaneously. Instead, the decision process can be thought of as being composed of two stages. In the first stage, agents maximize their income by optimally choosing the division of time between education and work. In the second stage, optimal consumption is determined. This decision is independent from the actual flow of income but only depends on its net present value at time zero.

Proposition 2. *Let Assumption A0 hold, $u^*(\cdot)$ be an optimal control in the education problem and $h^*(\cdot)$ be the trajectory corresponding to $u^*(\cdot)$. Then there exists an adjoint variable $\xi(\cdot)$ defined as*

$$\xi(t) = \int_t^\infty e^{-r(s,0)} w(s) u^*(s) e^{-\chi \int_t^s [1-u^*(\tau)]\,\mathrm{d}\tau}\,\mathrm{d}s \tag{14}$$

that satisfies almost everywhere the adjoint equation

$$-\dot{\xi}(t) = \chi[1-u^*(t)]\xi(t) + e^{-r(t,0)} w(t) u^*(t). \tag{15}$$

The optimal control $u^*(t)$ maximizes the Hamiltonian

$$\mathcal{H}(h,u,\xi,t) = e^{-r(t,0)} w(t) h u + \chi \xi h (1-u)$$

evaluated along the optimal path $h = h^*(t)$, $\xi = \xi(t)$ over all admissible $u \in [0,1]$ for almost all $t \in [0,\infty)$. As a consequence,

$$u^*(t) = \begin{cases} 1, & \textit{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) > 0, \\ 0, & \textit{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) < 0, \\ \textit{singular}, & \textit{if} \quad e^{-r(t,0)} w(t) - \chi\xi(t) = 0. \end{cases} \tag{16}$$

Proof. As the integrand in the definition of $\xi(t)$ in equation (14) is equal (up to a multiplicative constant) to the integrand in (8), the integral in (14) is absolutely convergent and finite. Thus, the statements of the proposition (in particular, the explicit representation of the adjoint variable) follow from Theorem 4.1 in Aseev and Veliov (2014). $\square$

4 Multiplicity of Optimal Solutions in the Partial Equilibrium Setting

Lucas (1988) shows that if a balanced growth path exists in the optimization problem proposed, in general equilibrium a unique and singular solution exists. We now analyze these assertions in the framework of partial equilibrium. We show that in partial equilibrium uniqueness of the solution is not given and instead infinitely many optimal solutions exist. As a side result, we obtain a constructive condition on the interest and wage variable for a singular solution to exist.

Theorem 1. *If an optimal balanced growth path exists for problem* (1) *subject to* (2)–(4) *with positive human capital growth, infinitely many optimal solutions for the education problem exist and can be characterized by*

$$\lim_{t\to\infty}\int_0^t u(s)\,\mathrm{d}s \to \infty. \tag{17}$$

Out of the infinitely many solutions that exist in partial equilibrium, the general equilibrium mechanism in Lucas (1988) identifies one of them. Other possible mechanisms to choose a unique solution are discussed in Section 5.

The proof of the theorem follows at the end of the section. We start by formulating one finding that is required for the proof in the form of a corollary.

Corollary 1. *If a singular solution exists for problem* (1) *subject to* (2)–(4)*, then*

$$\frac{\dot{w}(t)}{w(t)} = [r(t)-\chi], \quad t\in[0,\infty). \tag{18}$$

All admissible controls $u(t)$ that satisfy equation (17) *lead to the same value of the objective function in the education problem and the discounted future income is proportional to the initial level of human capital,*

$$\int_0^\infty e^{-r(t,0)}w(t)u(t)h(t)\,\mathrm{d}t = h(0)\frac{w(0)}{\chi}. \tag{19}$$

This implies that, for a singular solution in partial equilibrium with a constant wage rate per unit of effective labor and a constant interest rate, the interest rate must satisfy that $r(t)\equiv\chi$, while the wage per unit of effective labor may be any arbitrary positive constant. In general equilibrium, $r(t)=r=\chi$ typically defines a unique wage $w(t)=w(\chi)$ through the assumption of competitive factor markets.

The following theorem provides some special cases in which a balanced growth path exists in partial equilibrium.

Theorem 2. *Let the wage rate per unit of effective labor be constant and positive, $w(t)\equiv w>0$, and the interest rate be equal to the efficiency of human capital production, $r(t)\equiv\chi$, for all $t\in[0,\infty)$, and let additionally $\chi(1-\theta)<\rho$ and $0<(\chi-\rho)/(\chi\theta)<1$. Then a balanced growth path exists.*

Note that, due to the partial equilibrium setting, the assertion of the theorem is independent of $k(0)$ and $h(0)$. This implies that for two agents with different initial human capital $h_1(0)\neq h_2(0)$, both choosing the balanced growth path solution, the difference $h_1(t)-h_2(t)$ grows exponentially over time, but the fraction $h_1(t)/h_2(t)$ remains constant. The same holds true for physical capital.

Proof of Theorem 1. We prove Theorem 1 in three steps. First, we show that the existence of an optimal balanced growth path with positive human capital growth implies the existence of a singular solution. Then, we derive a necessary condition for an optimal solution to exist. Finally, we show that infinitely many optimal solutions exist.

The assumption of an optimal balanced growth path with positive human capital growth implies that for some constant $d > 0$, it holds that $d = \dot{h(t)}/h(t) = \chi[1 - u^*(t)]$. Thus, $u^*(t)$ is constant itself, $u^*(t) \equiv 1 - d/\chi < 1$, $t \in [0, \infty)$. The boundary value $u^*(t) \equiv 0$ has objective value zero in the education problem, which is clearly not optimal because controls that lead to a positive objective value exist. Therefore, $0 < u^*(t) < 1$ for almost all $t \in [0, \infty)$ which proves that a singular optimal solution exists.

Next, we derive the necessary condition for a singular solution presented in Corollary 1 by using the optimality conditions of Proposition 2. Consider the switching function $\zeta(\cdot)$, defined as

$$\zeta(t) = e^{-r(t,0)} w(t) - \chi \xi(t),$$

with a derivative over time given by

$$\begin{aligned} \dot{\zeta}(t) &= -r(t) e^{-r(t,0)} w(t) + e^{-r(t,0)} \dot{w}(t) - \chi \dot{\xi}(t) = \\ &= -r(t) e^{-r(t,0)} w(t) + e^{-r(t,0)} \dot{w}(t) + \chi[1 - u^*(t)] e^{-r(t,0)} w(t) + \chi e^{-r(t,0)} w(t) u^*(t), \end{aligned}$$

where the latter equality holds due to the adjoint equation (15). If a control is singular over the interval $[t_1, t_2) \subset [0, \infty)$, then the switching function $\zeta(t)$ is zero, thus

$$\chi \xi(t) = e^{-r(t,0)} w(t).$$

Therefore,

$$\dot{\zeta}(t) = e^{-r(t,0)} w(t) \left\{ \frac{\dot{w}(t)}{w(t)} - [r(t) - \chi] \right\} \tag{20}$$

for all $t \in [t_1, t_2)$. The existence of a singular solution implies $\dot{\zeta}(t) = 0$ and therefore the condition given by equation (18).

Note that the conditions $\zeta(t) = 0$ and $\dot{\zeta}(t) = 0$ are independent of the control $u(t)$. If these conditions are fulfilled, every control $u(t)$ satisfies the necessary optimality conditions and therefore infinitely many solutions and infinitely many singular solutions exist. We now show that all of them are optimal as well.

Considering any control $u(t)$, the objective value of the education problem is given by

$$\begin{aligned}
&\int_0^\infty e^{-r(t,0)} w(t)u(t)h(t)\,\mathrm{d}t = \\
=&\int_0^\infty e^{-r(t,0)} e^{\chi t} w(t)u(t)h(0)e^{-\chi\int_0^t u(s)\,\mathrm{d}s}\,\mathrm{d}t = \\
=&e^{-r(t,0)} e^{\chi t} w(t)\frac{h(0)}{\chi} e^{-\chi\int_0^t u(s)\,\mathrm{d}s}\Big|_0^\infty - \\
&-\int_0^\infty [\dot{w}(t) + \chi w(t) - r(t)w(t)] e^{-r(t,0)} e^{\chi t} h(0) e^{-\chi\int_0^t u(s)\,\mathrm{d}s}\,\mathrm{d}t.
\end{aligned}$$

Since the wage rate satisfies (18), the integrand of the second term is zero. Equation (18) further implies that $w(t) = w(0)e^{r(t,0)}e^{-\chi t}$, which leads to the first term being equal to

$$\frac{w(0)h(0)}{\chi}\left[1 - e^{-\chi \lim_{t\to\infty}\int_0^t u(s)\,\mathrm{d}s}\right].$$

The objective value is thus maximized and equal to $w(0)h(0)/\chi$ for any control $u(t)$ satisfying equation (17). This equation is in turn satisfied by infinitely many solutions, which proves the claim of the theorem. Note that it follows that along every optimal control, discounted future income is equal to the initial level human capital multiplied by $w(t)/\chi$, that is, the expression given by equation (19). ☐

Proof of Theorem 2. If $r(t)$ and $w(t)$ are constant, it follows that $r(t) \equiv \chi$, $t \in [0, \infty)$, while the wage rate per unit of effective labor is undetermined, $w(t) \equiv w > 0$.

Consider the maximal achievable income $I(t) := \chi k(t) + wh(t)$, which is equal to the income at time t under constant interest rates and wages if the agent works full time. Taking the derivative of $I(t)$ with respect to time, we obtain

$$\begin{aligned}
\dot{I}(t) &= \chi\dot{k}(t) + w\dot{h}(t) = \\
&= \chi[\chi k(t) + wu(t)h(t) - c(t)] + w\{h(t)\chi[1 - u(t)]\} = \\
&= \chi[I(t) - c(t)].
\end{aligned}$$

Recalling the fact pointed out in the proof of Theorem 1 that discounted future income equals $wh(t)/\chi$, independently of the control $u(t)$, optimal consumption $c^*(t)$ can be expressed as

$$c^*(t) = \frac{k(t) + \int_t^\infty e^{-r(s,t)} wu(s)h(s)\,\mathrm{d}s}{\int_t^\infty e^{-r(s,t)} e^{\frac{r(s,t)-\rho(s-t)}{\theta}}\,\mathrm{d}s} = [\chi k(t) + wh(t)]\left[\frac{\rho - \chi(1-\theta)}{\theta\chi}\right].$$

This expression follows from equation (13) by means of a time shift. Exchanging 0 in equation (13) by t and t in equation (13) by s and inserting this representation of $c^*(t)$ in the equation for $\dot{I}(t)$, we obtain that the maximal achievable income grows at a constant rate, independently of the control $u(t)$,

$$\frac{\dot{I}(t)}{I(t)} = \frac{\chi - \rho}{\theta}.$$

Due to our assumption $0 < (\chi - \rho)/(\chi\theta) < 1$, there exists $u \in (0,1)$ such that $\dot{h(t)}/h(t) = \chi(1-u) = (\chi - \rho)/\theta$. Inserting this u and the expression for $c^*(t)$ in the equation for $\dot{k}(t)$, we obtain

$$\frac{\dot{k}(t)}{k(t)} = \frac{\chi - \rho}{\theta}.$$

Due to Corollary 1, the control $u(t) \equiv u$ is also optimal, which concludes our proof of the existence of a balanced growth path. $\square$

5 Solution Selection Mechanisms in the Lucas Model

In this section we discuss some approaches serving to select a unique optimal solution. Our results for the partial equilibrium formulation of the Lucas model show that agents have infinitely many optimal choices for the control $u(t)$. Among the several solution selection mechanisms available to overcome this indeterminacy we highlight two of them which introduce nonlinearity in the Hamiltonian of the education problem: the *general equilibrium* formulation of the model (as in Lucas, 1988) and *nonlinearity in the human capital accumulation* function.

If the interest rate and the wage rate are determined endogenously in a general equilibrium setting, the problem of a representative agent has a unique solution, as it is shown in the original paper by Lucas (1988). This is the case because labor supply is increasing with respect to the wage rate, while labor demand is decreasing. Consequently, there exists a unique intersection of labor demand and labor supply at the unique equilibrium wage. If the representative agent decreases (increases) her labor supply, wages rise (fall) and agents have incentives to increase (decrease) labor supply accordingly. Thus, the general equilibrium framework acts as a "selector" of a unique optimal choice of control.

Analytically, for a given control $u(t)$, a production function $Y(t) = F[K(t), L(t)]$ and the assumption that the wage equals the marginal productivity of labor, it holds that

$$w[t, u(t)] = \frac{\partial F[K(t), L(t)]}{\partial L(t)}|_{[K(t),L(t)]=[K(t),u(t)H(t)]}.$$

Inserting this expression into the Hamiltonian of the *education problem*, the optimal control $u^*(t)$ maximizes

$$\mathcal{H}[t, h, u, \xi] = w[t, u(t)]u(t)h(t)e^{-r(t,0)} + \chi\xi(t)h(t)[1 - u(t)]$$

over the set $u(t) \in [0, 1]$. If $w[t, u(t)]u(t) + \chi\xi(t)e^{r(t,0)}$ is a concave function in $u(t)$, then a unique maximum $u^*(t)$ exists. This holds true, for example, for a Cobb-Douglas production function (as used in the original Lucas model). In the Cobb-Douglas case, the unique optimal control is given by

$$u^*(t) = \begin{cases} \left[\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2k(t)^\alpha}\right]^{\frac{-1}{\alpha}}, & \text{if } \left[\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2k(t)^\alpha}\right]^{\frac{-1}{\alpha}} \le 1, \\ 1, & \text{if } \left[\frac{\xi(t)\chi e^{r(t,0)}}{h(t)^{1-\alpha}A(1-\alpha)^2k(t)^\alpha}\right]^{\frac{-1}{\alpha}} > 1. \end{cases}$$

A second mechanism that acts as a solution selector can be conceptualized by including a non-linearity in the human capital production. In the partial equilibrium setting, the linearity of $\dot{h}(t)$ and $\dot{k}(t)$ with respect to $u(t)$ and $h(t)$, together with the equality of marginal returns in equilibrium, implies that agents are indifferent between all controls $u(t)$ that result in the same amount of working time. Thus, switching back and forth between education and work becomes also an optimal behavior for individuals in the partial equilibrium formulation of the Lucas model. Introducing a nonlinearity with respect to $u(t)$ in the human capital accumulation equation is another possible way to achieve uniqueness in the optimal control of the partial equilibrium problem.

In the more general formulation of the Lucas-Uzawa model in Uzawa (1965), the following functional form is proposed for the dynamics of human capital,

$$\dot{h}(t) = h(t)G[1 - u(t)],$$

where $G(\cdot)$ is an increasing function with $G(0) = 0$, non-negative first derivative and non-positive second derivative. Furthermore, $G(1)$ is assumed not to be very large compared to the discount rate (which translates into an assumption on the relations between χ, ρ, and θ in our formulation of the problem) and $G(0) + G'(0)$ is assumed sufficiently large.

Replacing the linearity assumption implied by the human capital accumulation equation (3)in our analysis by $G[1 - u(t)] = \chi[1 - u(t)]^\nu$ with $\nu < 1$, the Hamiltonian of the *education problem* becomes

$$\mathcal{H}[t, h, u, \xi] = e^{-r(t,0)}w(t)u(t)h(t) + \xi\chi[1 - u(t)]^\nu,$$

and its first order condition with respect to $u(t)$ is given by

$$e^{-r(t,0)}w(t)h(t) - \xi(t)\nu\chi h(t)[1 - u(t)]^{\nu-1} = 0.$$

The boundary solution $u(t) = 1$ is not optimal since $e^{-r(t,0)}w(t)h(t)$ is strictly positive for all t. The assumption $\nu < 1$ implies that $u^*(t)$ is uniquely determined by

$$u^*(t) = \begin{cases} 0, & \text{if } 1 - \left[\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right]^{\frac{1}{\nu-1}} < 0, \\ 1 - \left[\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right]^{\frac{1}{\nu-1}}, & \text{if } 1 - \left[\frac{e^{-r(t,0)}w(t)h(t)}{\xi(t)\nu\chi h(t)}\right]^{\frac{1}{\nu-1}} \geq 0. \end{cases}$$

6 Conclusions and paths of further research

We show that reformulating the Lucas model in the partial equilibrium setting provides new insights to the optimal control problem that economic agents face in the original specification. In particular, we show that the partial equilibrium reformulation of the optimization problem in Lucas (1988) exhibits multiple optimal solutions. It may be optimal for the representative agent to allocate her whole available time to education at the beginning of her life and to focus on labor supply only at later periods. Even switching back and forth between full-time education and full-time work is a feasible and optimal solution for rational individuals. We present two equilibrium selection mechanisms that are based on introducing nonlinearity into the optimization problem: assuming a general equilibrium setting or assuming nonlinearities in human capital accumulation.

Our analysis also provides additional results concerning the equivalence of different formulations of the partial equilibrium Lucas model. Conditional on given paths of interest rates and wages, we show that the optimization problem in Lucas (1988) can be rewritten as a two-stage optimization problem. In the first stage, agents solve the problem of optimally choosing the division of time between human capital accumulation and production so as to maximize lifetime income. In the second stage, agents distribute consumption optimally over their infinite lives, given lifetime income.

Our results show that if a balanced growth path exists, all solutions that satisfy $\lim_{t\to\infty} \int_0^t u(s)\,\mathrm{d}s \to \infty$ are optimal. Thus, all of the following stylized categories of agent behavior are optimal in the framework of the Lucas model in the partial equilibrium setting:

- No Learning: Working full time without additional education, $u(t) \equiv 1$ for all $t \in [0, \infty)$.
- Continuous Learning: Working part time and educating the rest of the time $0 < u(t) \equiv u < 1$ for all $t \in [0, \infty)$.
- Educational Leave: Switching back and forth between work and education, e.g. $u(t) \equiv 1$ for $t \in [2n, 2n+1)$ and $u(t) \equiv 0$ for $t \in [2n-1, 2n)$ and $n \in \mathbb{N}$.
- Standard Schooling: Full-time education, $u(t) \equiv 0$, for $t \in [0, s]$ and full-time work, $u(t) \equiv 1$, for $t \in (s, \infty)$.

The assumptions for the equivalence are the existence of a singular solution, the linearity of the human capital dynamics and the infinite lifetime of the agents, or, in the dynastic interpretation, that human capital is fully inherited. If human capital accumulation was less costly for younger individuals than older ones, it is plausible that full time education is optimal in younger years, perhaps followed by a period in which partial education (either continuous learning or educational leave) is optimal, and finally older individuals would work full time. In case of a finite lifetime without passing the acquired human capital to children, we would expect similarly a decline in education at older ages. Altogether, this implies that such a framework is able to deliver realistic age-education profiles. Non-linear human capital production would shift the optimal solution either to continuous learning (in case of human capital production being concave in time devoted for learning) or to educational leave (convexity in time devoted for learning). Such richness of results for human capital dynamics makes the reformulation of the model put forward in this study particularly promising for further research on optimal education paths in economic growth models.

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