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A NOTE ON QUEUEING SYSTEMS EXPOSED TO DISASTERS

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ABSTRACT. We discuss queueing systems subject to total disasters. If the time intervals between successive disasters are i.i.d. random variables independent of arrival and service process and arrivals form a Poisson process, then the transient and the asymptotic analysis of such models may be based on Feller's Second Renewal Theorem. Several examples are given: the limiting behavior of $M/G/1$ in case of exponential disasters and its special cases $M/M/1$, $M/M/1/K$ and $M/M/\infty$.

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1. Introduction

In this paper we consider queueing systems which are exposed to disasters, a disaster being a random event resulting in the extinction of all customers in the system. Technically speaking the sample paths of the queueing process collapse from time to time, the process is pushed back to the origin and starts anew. Such models have been discussed recently by Di Crescenzo, Giorno, Nobile and Ricciardi (2003) and Krishna Kumar and Arivudainambi (2000). In these papers transient and steady state results for the $M/M/1$ system are given when the time intervals between disasters have an exponential distribution. The results are of considerable interest in the analysis of computer- and communication systems where disasters caused by external effects have an important influence on costs and performance. Di Crescenzo et al. (2003) report also about an interesting application is biology. In Boxma, Perry and Stadje (2001) the workload process of $M/G/1$ systems in presence of disasters is analyzed. In this paper alternative scenarios of the disaster process are also considered, in particular disasters which are bound to boundary crossing events of the workload. A further reference of more technical character is Grigorescu and Kang (2003). A. Krinik and C. Mortensen (2007) based their analysis of finite Markovian systems on the Cayley-Hamilton theorem.

In the queueing systems we are going to discuss customers arrive according to a Poisson process with rate λ, there are one or more parallel service channels, the process of service being independent of the arrival process. The system is exposed to total disasters, i.e. at random times catastrophes occurs which lead to the complete extinction of all customers present in the system at that time. The service facility is shut down and restarts immediately, waiting for the next customer to arrive. We assume that the time intervals between successive disasters $D_i, i = 1, 2, \dots$ are i.i.d. random variables with non-arithmetic distribution $dF(t)$ and finite expectation $E(D)$. Disasters are assumed to be independent of arrival and service process. They may also occur at a time when the queueing system is empty, in which case they clearly have no effect on the number of customers in the system but still may have an effect on the costs running the system, since a disaster results in a shutdown and restart of the service facility[1].

Let us fix notation first. By $Q(t)$ we denote the number of customers in the system at time t. We assume that at time $t = 0$ a disaster occurred, so that $Q(0) = 0$. Furthermore define an auxiliary queueing process $Q^0(t)$ which behaves exactly like $Q(t)$ with the exception, that $Q^0(t)$ is not exposed to disasters.

Furthermore define

$$\begin{aligned} p_0(t,k) &= P(Q^0(t) = k | Q^0(0) = 0) \\ q(t,k) &= P(Q(t) = k, T_1 > t | Q(0) = 0) = p_0(t,k)(1 - F(t)) \\ p(t,k) &= P(Q(t) = k | Q(0) = 0) \end{aligned}$$

There is an intimate connection between these functions, since the times when disasters occur form a renewal sequence and the arrival process is Markovian: whenever a disasters occurs the process $Q(t)$ starts from scratch independent of its past. It is therefore a natural candidate of

Theorem 1.1 (Feller's Second Renewal Theorem)**.** *Under the conditions state above*

$$p(t,k) = q(t,k) + \int_0^t p(t-u,k)\,dF(u) \tag{1}$$

Moreover the functions $p(t,k)$ converge to a limit as $t \to \infty$:

$$\pi_k = \lim_{t\to\infty} p(t,k) = \frac{1}{E(D)} \int_0^\infty q(t,k)\,dt \tag{2}$$

This theorem and its proof are given in Feller (1971, pp. 379).

A formal solution of (1) may be obtained in a rather routine manner in terms of Laplace-Stieltjes transforms (LST). For this purpose let $p^*(s,k), q^*(s,k)$ and $f^*(s)$ denote the LST of $p(t,k), q(t,k)$ and $dF(t)$, respectively. Then from (1):

$$p^*(s,k) = \frac{q^*(s,k)}{1 - f^*(s)} \tag{3}$$

[1]Interestingly Di Crescenzo et al. (2003) assume explicitly that disasters may occur only when the system is busy. This condition is not necessary in case of exponential disasters due to the Markov property of the exponential distribution. In that paper the reader will find also the density of the time τ until the occurrence of the first disaster, an interesting and non trivial result. It turns out that this density is not exponential, which is not surprising. In our framework τ is just the time when a disaster hits a *non empty* system for the first time.

Whether this formula is useful after all, clearly depends on our ability to find the transform $q^*(s,k)$ and to invert (3). Thus nice, closed form expressions of the functions $p(t,k)$ can be expected in exceptional cases only. This is true also for (2), which still requires an expression for $p_0(t,k)$. Unfortunately such formulas are rarely available.

However, there is one notable exception:

Theorem 1.2 (Exponential disasters). *Suppose that the time intervals between successive disasters are exponentially distributed with mean $1/\gamma$. Then the following integral representation of the functions $p(t,k)$ holds:*

$$p(t,k) = e^{-\gamma t} p_0(t,k) + \gamma \int_0^t e^{-\gamma t} p_0(u,k)\,du. \tag{4}$$

The limit $t \to \infty$ in (4) exists and is given by

$$\pi_k = \lim_{t\to\infty} p(t,k) = \gamma \int_0^\infty e^{-\gamma t} p_0(t,k)\,dt = \gamma\, p_0^*(\gamma,k), \tag{5}$$

where $p_0^*(s,k)$ is the LST of $p_0(t,k)$ evaluated at $s=\gamma$.

Proof. We have

$$f^*(s) = \frac{\gamma}{s+\gamma} \qquad \Longrightarrow \frac{1}{1-f^*(s)} = 1 + \frac{\gamma}{s}$$

Thus

$$q^*(s,k) = \int_0^\infty e^{-st} p_0(t,k) e^{-\gamma t}\,dt = p_0^*(s+\gamma,k)$$

It follows that

$$p^*(s,k) = p_0^*(s+\gamma,k) + \frac{\gamma}{s} p_0^*(s+\gamma,k) \tag{6}$$

The formal inversion of (6) is easy and yields (4). □

In the sequel we present various applications of the results of this section.

2. The limiting distribution of $M/G/1$ systems with exponential disasters

The system $M/G/1$ has be studied extensively, therefore many results, mostly in terms of LST and generating functions, are available. A good source is the book of Prabhu (1965).

Assume that service times have distribution $dB(t)$ with LST $\psi(s)$ and let $\eta(s)$ satisfy the functional equation

$$\eta(s) = \lambda + s - \lambda\psi(\eta)$$

It is well know that this root plays a central role in the transient analysis of the system $M/G/1$. And it is the determination of this root, which makes the analysis of $M/G/1$ rather difficult. However, for the asymptotic analysis of this model we only require $\eta_\gamma := \eta(\gamma)$, which can be determined by a simple numerical root finding procedure.

Let us now determine the limiting distribution π_k. Since for the undisturbed queueing process $Q^0(t)$ the probability of an empty system has LST (see Prabhu (1965, pp. 77))

$$p_0^*(s,0) = \frac{1}{\eta(s)},$$

we have by (4):

$$\pi_0 = \frac{\gamma}{\eta_\gamma} \tag{7}$$

For $k \geq 1$ define the generating function (Prabhu (1965, pp. 85)):

$$H(\gamma, z) = \sum_{k\geq 1} z^k p_0^*(\gamma, k) = \lambda z \frac{1-\psi(\gamma+\lambda-\lambda z)}{z-\psi(\gamma+\lambda-\lambda z)} \frac{z-\psi(\eta_\gamma)}{(\gamma+\lambda-\lambda z)\eta_\gamma}$$

Again by (4) $\pi_k = \gamma[z^k]H(z,\gamma)$. In the case of rational $\psi(s)$ the generating function $H(z,\gamma)$ is rational too and the limiting probabilities π_k may be obtained by a decomposition into partial fractions.

As a particular example consider the classical $M/M/1$ system with service times having mean $1/\mu$. Here

$$\psi(s) = \frac{\mu}{\mu+s}$$

and the root $\eta(s)$ can be given explicitly:

$$\begin{aligned}\eta(s) &= \frac{s+\lambda-\mu-\sqrt{(s+\lambda-\mu)^2+4\mu s}}{2}\\ &= \frac{s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}{2} - \mu\end{aligned}$$

We require η_γ, which can be written more conveniently as

$$\eta_\gamma = \frac{\lambda}{v} - \mu, \qquad v = \frac{\lambda+\mu+\gamma-\sqrt{(\lambda+\mu+\gamma)^2-4\lambda\mu}}{2\mu} \tag{8}$$

Therefore, because $\lambda/v + \mu v = \lambda+\mu+\gamma$:

$$\pi_0 = \frac{\gamma}{\lambda/v-\mu} = 1-v$$

Furthermore, using $\psi(\eta_\gamma) = \mu v/\lambda$, the generating function $H(z,\gamma)$ turns out to be

$$H(z,\gamma) = -\frac{1}{\eta_\gamma} \cdot \frac{\lambda z(z-\mu v/\lambda)}{\lambda z^2 - (\lambda+\mu+\gamma)z+\mu}$$

The denominator factors nicely into

$$\lambda\left(z-\frac{1}{v}\right)\left(z-\frac{\mu v}{\lambda}\right)$$

Hence

$$H(z,\gamma) = -\frac{1}{\eta_\gamma}\cdot\frac{z}{z-1/v} = \frac{1}{\eta_\gamma}\cdot\frac{vz}{1-vz}$$

It follows that for $k \geq 1$:

$$\pi_k = [z^k]\frac{\gamma}{\eta_\gamma}\cdot\frac{vz}{1-vz} = (1-v)v^k \tag{9}$$

This is the result of Krishna Kumar and Arivudainambi (2000). An alternative derivation of (9) will be given below.

The busy period. The occurrence of disasters has rather drastic effects on the recurrence properties of the queue-length process. We may get an impression of these effects when we consider for instance the duration of a busy period initiated by a single customer:

$$T_1 = \inf\{t : t > 0, Q(t) = 0, Q(0) = 1\},$$

and let T_1^0 denote the corresponding random variable for the undisturbed process $Q^0(t)$. The LST of T_1^0 is given by $\Gamma(s) = \psi(\eta)$ (Prabhu (1965, p. 75)). Now in a system exposed to disasters a busy period not only terminates when the system runs out of customers due to the service process but also when customers are killed by a disaster. If disasters are exponential, then

$$P(T_1 > t) = P(T_1^0 > t, D_1 > t) = e^{-\gamma t}P(T_1^0 > t)$$

The corresponding LST is found to be

$$\begin{aligned} b(s) &= \int_0^\infty e^{-st}\int_t^\infty e^{-\gamma u}P(T_1^0 > u)du \\ &= \frac{1-\psi(\eta(s+\gamma))}{s+\gamma} \end{aligned} \tag{10}$$

It follows that

$$E(T_1) = \lim_{s\to 0} b(s) = \frac{1-\psi(\eta_\gamma)}{\gamma}, \tag{11}$$

which is always finite and less than $1/\gamma$, the mean length of the intervals between disasters. In the case of simple $M/M/1$ we obtain the neat formula

$$E(T_1) = \frac{1}{\gamma}\left(1-\frac{\mu v}{\lambda}\right) = \frac{1}{\lambda}\frac{v}{1-v}, \tag{12}$$

which may be found also in Di Crescenzo et al. (2003).

3. The system $M/M/1/K$, transient results

Even when disasters are exponential, transient results are rather difficult to come by. However, the system $M/M/1/K$ is an exception worth to consider. This model is appealing not only because its time dependent distribution $p_0(t,k)$ is simple, in fact these functions are merely linear combinations of exponential functions: by the classical formula of Morse (1958)

$$p_0(t,k) = \pi_k^0 + \frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{b_\nu}e^{-b_\nu t}, \tag{13}$$

where as usual λ and μ denote arrival and service rate, $\rho = \lambda/\mu$ and

$$\begin{aligned} a_{k,\nu} &= \sin\theta_\nu\left(\sin(k+1)\theta_\nu - \rho^{-1/2}\sin k\theta_\nu\right)\\ b_\nu &= \lambda+\mu-2\sqrt{\lambda\mu}\cos\theta_\nu\\ \theta_\nu &= \frac{\nu\pi}{K+1}, \qquad \nu = 1,2,\ldots,K \end{aligned}$$

and π_k^0 are the steady state probabilities given by

$$\pi_k^0 = \begin{cases} \dfrac{(1-\rho)\rho^k}{1-\rho^{K+1}} & \text{if } \rho \neq 1\\ \dfrac{1}{K+1} & \text{if } \rho = 1 \end{cases}$$

If disasters are exponential with expectation $1/\gamma$, then by (4):

$$\begin{aligned} p(t,k) &= e^{-\gamma t}p_0(t,k) + \gamma\int_0^t e^{-\gamma u}p_0(u,k)\,du\\ &= \pi_k^0 + \frac{2\lambda\gamma\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{b_\nu(\gamma+b_\nu)} + \frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{k}\frac{a_{k,\nu}}{\gamma+b_\nu}e^{-(\gamma+b_\nu)t} \end{aligned} \tag{14}$$

Observe that the second term in (14) is independent of t, thus the state probabilities $p(t,k)$ split nicely into a steady state part and a time dependent part. In fact, passing to the limit $t\to\infty$ in (14), we find

$$\pi_k = \pi_k^0 + \frac{2\lambda\gamma\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{b_\nu(\gamma+b_\nu)} \tag{15}$$

which demonstrates that the steady state distribution of the system in presence of disasters is given by the steady state distribution of the undisturbed system plus an additional term taking care of the disasters.

If disasters have a more general distribution $dF(t)$ other than exponential with LST $f^*(s)$, then $q(t,k) = p_0(t,k)(1-F(t))$ and by the shift property of LSTs:

$$q^*(s,k) = \pi_k^0 \frac{1-f^*(s)}{s} + \frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{b_\nu}\,\frac{1-f^*(s+b_\nu)}{s+b_\nu} \tag{16}$$

but the general formula (3) is helpful in very special cases only. For instance, if $f^*(s)$ is rational, then so is $(1-f^*(s))^{-1}$ and a decomposition into partial fractions may be applied. Still nice formulas may be obtained for the asymptotic distribution. Taking the limit $s \to 0$ in (16) we get

$$\begin{aligned}\lim_{s\to 0} q^*(s,k) &= \int_0^\infty q(t,k)dt \\ &= \pi_k^0\, E(D) + \frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{b_\nu}\,\frac{1-f^*(b_\nu)}{b_\nu},\end{aligned}$$

and by (2):

$$\pi_k = \pi_k^0 + \frac{1}{E(D)}\cdot\frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}(1-f^*(b_\nu))}{b_\nu^2} \tag{17}$$

$M/M/1$ **as limiting case of** $M/M/1/K$. If we put $\Delta x = 1/K+1$ and let $K \to \infty$, then both summations in (14) converge to Riemann integrals. For the first sum we have:

$$\frac{2\lambda\gamma\rho^{k/2}}{K+1}\sum_{a_{k,\nu}} b_\nu(\gamma+b_\nu) \quad\to\quad \frac{2\lambda\gamma\rho^{k/2}}{\pi}\int_0^\pi \frac{\alpha_k(x)}{\beta(x)(\gamma+\beta(x))}dx, \tag{18}$$

where

$$\alpha_k(x) = \sin x\left(\sin(k+1)x - \rho^{-1/2}\sin kx\right) \tag{19}$$

$$\beta(x) = \lambda+\mu-2\sqrt{\lambda\mu}\cos x \tag{20}$$

The integral in (18) can be evaluated explicitly. To see this, observe that the integrand is an even function of x. Thus the integral may be written as:

$$I = \frac{\lambda\gamma\rho^{k/2}}{\pi}\int_0^{2\pi}\frac{\sin x\left[\sin((k+1)x)-\rho^{-1/2}\sin(kx)\right]}{(\lambda+\mu-2\sqrt{\lambda\mu}\cos x)(\lambda+\mu+\gamma-2\sqrt{\lambda\mu}\cos x)}\,dx$$

Now put $z = e^{xi}$, so that I becomes by this substitution:

$$\begin{aligned} I &= -\frac{\gamma\rho^{k/2}}{4\pi i\mu}\oint\frac{(z^2-1)\left[z^{2k+1}(z-\rho^{-1/2})+\rho^{-1/2}(z-\rho^{1/2})\right]}{(z^2-\frac{\lambda+\mu}{\sqrt{\lambda\mu}}z+1)(z^2-\frac{\lambda+\mu+\gamma}{\sqrt{\lambda\mu}}z+1)}\,\frac{dz}{z^{k+1}} \\ &= -\frac{\gamma\rho^{k/2}}{4\pi i\mu}\oint\frac{(z^2-1)\left[z^{2k+1}(z-\rho^{-1/2})+\rho^{-1/2}(z-\rho^{1/2})\right]}{(z-\rho^{1/2})(z-\rho^{-1/2})(z-w)(z-1/w)}\,\frac{dz}{z^{k+1}},\end{aligned}$$

where w is given by

$$w = \frac{\lambda + \mu + \gamma - \sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu}}{2\sqrt{\lambda}\mu},$$

and the contour of integration is the unit circle.

As can be seen by inspection the integrand has single poles at $z = \rho^{1/2}$, $z = \rho^{-1/2}$, $z = w$ and $z = 1/w$. Furthermore there is a pole of order $k + 1$ at $z = 0$. Let us find the residues at these poles and note that we do not need the residue at $z = 1/w$, since $w < 1$ and therefore $1/w$ will always lie outside the unit circle.

We obtain for $z = \rho^{1/2}$:

$$R_1 = -\frac{1}{4\pi i}(1 - \rho)\rho^k,$$

For $z = \rho^{-1/2}$:

$$R_1' = \frac{1}{4\pi i}(1 - \rho)\rho^k.$$

And finally for $z = w$:

$$R_2 = -\frac{1}{4\pi i}\left[\rho^{\frac{k+1}{2}} w^{k+1} - \rho^{\frac{k}{2}} w^k - \rho^{\frac{k}{2}} w^{-k} + \rho^{\frac{k+1}{2}} w^{-k-1}\right].$$

Let us put $v = \rho^{1/2} w$ in R_2, where v is given by (8):

$$R_2 = \frac{1}{4\pi i}\left[(1 - v)v^k + \left(1 - \frac{\rho}{v}\right)\left(\frac{\rho}{v}\right)^k\right].$$

The residue R_3 at $z = 0$ requires more work to be done. Since the pole is of order $k + 1$, R_3 turns out to be

$$R_3 = -\frac{\gamma \rho^{k/2}}{4\pi i \mu}[z^k]g(z),$$

where $[z^k]$ denotes the familiar coefficient operator and $g(z)$ is given by

$$\begin{aligned} g(z) &= \frac{(z^2 - 1)\left[z^{2k+1}(z - \rho^{-1/2}) + \rho^{-1/2}(z - \rho^{1/2})\right]}{(z - \rho^{1/2})(z - \rho^{-1/2})(z - w)(z - 1/w)} \\ &= \left[z^{2k+1}(z - \rho^{-1/2}) + \rho^{-1/2}(z - \rho^{1/2})\right] h(z), \end{aligned}$$

with

$$h(z) = \frac{(z^2 - 1)}{(z - \rho^{1/2})(z - \rho^{-1/2})(z - w)(z - 1/w)}.$$

It is easy to expand $h(z)$ into partial fractions:

$$h(z) = \frac{\sqrt{\lambda\mu}}{\gamma}\left[-\frac{1}{z - \rho^{1/2}} - \frac{1}{z - \rho^{-1/2}} + \frac{1}{z - w} + \frac{1}{z - 1/w}\right].$$

Thus

$$[z^m]h(z) = \frac{\sqrt{\lambda\mu}}{\gamma}\left[\rho^{-\frac{m+1}{2}} + \rho^{\frac{m+1}{2}} - w^{-m-1} - w^{m+1}\right].$$

It follows that

$$R_3 = -\frac{1}{4\pi i}\left[(1-\rho)\rho^k - (1-v)v^k - \left(1-\frac{\rho}{v}\right)\left(\frac{\rho}{v}\right)^k\right].$$

Assume now that $\rho < 1$. Then by the residue theorem we obtain

$$I = 2\pi i(R_1 + R_2 + R_3) = (1-v)v^k - (1-\rho)\rho^k.$$

On the other hand, when $\rho > 1$, then

$$I = 2\pi i(R_1' + R_2 + R_3) = (1-v)v^k.$$

In the case $\rho = 1$, the integrand has a pole of order 2 at $z = 1$, but this singularity is removable and we find that the above formula holds also when $\rho = 1$. Thus we have finally proved that

$$\frac{2\lambda\gamma\rho^{k/2}}{\pi}\int_0^\pi \frac{\alpha_k(x)}{\beta(x)(\gamma+\beta(x))}dx = \begin{cases} (1-v)v^k - (1-\rho)\rho^k & \rho < 1 \\ (1-v)v^k & \rho \geq 1 \end{cases}.$$

Similarly for the second summation in (14):

$$\frac{2\lambda\rho^{k/2}}{K+1}\sum_{\nu=1}^{K}\frac{a_{k,\nu}}{\gamma+b_\nu}e^{-(\gamma+b_\nu)t} \quad \to \quad \frac{2\lambda\rho^{k/2}}{\pi}\int_0^\pi \frac{\alpha_k(x)}{\gamma+\beta(x)}e^{-(\gamma+\beta(x))t}dx$$

Putting things together we arrive at the following representation of the transient distribution of $M/M/1$ in case of exponential disasters:

$$p(t,k) = (1-v)v^k + \frac{2\lambda\rho^{k/2}}{\pi}\int_0^\pi \frac{\alpha_k(x)}{\gamma+\beta(x)}\,e^{-(\gamma+\beta(x))t}dx \tag{21}$$

Observe that also this formula splits nicely into a steady state part and a time dependent part. It should be noted, that (21) is an interesting alternative to the formulas given by Di Crescenzo et al. (2003) and Krishna Kumar and Arivudainambi (2000) in particular from a computational point of view. The computational burden to evaluate this integral is considerably less than the evaluation of an infinite sum of modified Bessel functions[2].

[2] Actually the integral in (21) may be evaluated as a sum of modified Bessel functions by using the substitution $x = e^{\theta i}$. The integrand then turns out to be proportional to $e^{\sqrt{\lambda\mu}(\theta+1/\theta)}$. This is just the generating function of the modified Bessel functions.

4. The system $M/M/\infty$ and exponential disasters

As a last application of Theorem 1.1 we discuss briefly the system $M/M/\infty$. We assume that disasters are exponential with mean $1/\gamma$. Arrivals occur with rate λ and mean service time is $1/\mu$. By Palm's famous formula (Palm (1943)):

$$p_0(t,k) = \frac{e^{-\rho}\rho^k}{k!} e^{\rho e^{-\mu t}} (1 - e^{-\mu t})^k, \qquad \rho = \lambda/\mu$$

Applying (4) leads directly to the following integral representation of the transient distribution:

$$p(t,k) = \frac{e^{-\rho}\rho^k}{k!} \left[e^{-\gamma t + \rho e^{-\mu t}} (1 - e^{-\mu t})^k + \gamma \int_0^t e^{-\gamma u + \rho e^{-\mu u}} (1 - e^{-\mu u})^k du \right] \tag{22}$$

And performing the passage to the limit $t \to \infty$ we obtain

$$\pi_k = \frac{e^{-\rho}\rho^k}{k!} \cdot \gamma \int_0^\infty e^{-\gamma t + \rho e^{-\mu t}} (1 - e^{-\mu t})^k dt$$

This integral is known (Gradshteyn and Ryzhik (1980, p. 308)), it can be written as product of a beta function and a confluent hypergeometric function:

$$\pi_k = \frac{e^{-\rho}\rho^k}{k!} \frac{\gamma}{\mu} B\left(\frac{\gamma}{\mu}, k+1\right) {}_1F_1\left(\frac{\gamma}{\mu}; \frac{\gamma}{\mu} + k + 1; \rho\right) \tag{23}$$

It is quite interesting to see how (23) factors: the first term is just the steady state distribution of $M/M/\infty$ without disasters whereas the remaining factors take care of the disaster process.

Formula (23) simplifies slightly if we express the beta function in terms of the gamma function:

$$\pi_k = \frac{e^{-\rho}\rho^k}{(\gamma/\mu)_k} {}_1F_1\left(\frac{\gamma}{\mu}; \frac{\gamma}{\mu} + k + 1; \rho\right) \tag{24}$$

where $(\gamma/\mu)_k$ is the Pochhammer symbol.

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