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A Kolmogorov-Smirnov Test for r Samples

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Abstract. We consider the problem of testing whether $r \geq 2$ samples are drawn from the same continuous distribution $F(x)$. The test statistic we will study in some detail is defined as the maximum of the circular differences of the empirical distribution functions, a generalization of the classical 2-sample Kolmogorov-Smirnov test to $r \geq 2$ independent samples. For the case of equal sample sizes we derive the exact null distribution by counting lattice paths confined to stay in the scaled alcove $\mathcal{A}_r$ of the affine Weyl group A_{r-1}. This is done using a generalization of the classical reflection principle. By a standard diffusion scaling we derive also the asymptotic distribution of the test statistic in terms of a multivariate Dirichlet series. When the sample sizes are not equal the reflection principle no longer works, but we are able to establish a weak convergence result even in this case showing that by a proper rescaling a test statistic based on a linear transformation of the circular differences of the empirical distribution functions has the same asymptotic distribution as the test statistic in the case of equal sample sizes.

Keywords: Kolmogorov-Smirnov test, lattice path counting, reflection principle, affine Weyl groups, asymptotics distrubution.

AMS Subject Classification: 05A15, 05A16, 62G10, 62G20

1. Introduction

In this paper we consider the problem of testing whether $r \geq 2$ samples are drawn from the same continuous distribution $F(x)$. We will be primarily concerned with the case that all samples are of equal size. Unequal sample sizes are dealt with in section 4. As a test statistic we will use the *circular differences*

$$\delta_r(n) = \max\left[\delta_{1,2}(n), \delta_{2,3}(n), \ldots, \delta_{r-1,r}(n), \delta_{r,1}(n)\right], \tag{1}$$

where $\delta_{ij}(n) = \sup_x[F_{n,i}(x) - F_{n,j}(x)]$, and $F_{n,i}(x), i = 1, 2, \ldots, r$ denote the empirical distribution functions of these samples. This problem has a long and fascinating history in mathematical statistics. In a classical paper Gnedenko and Korolyuk (1951) derived the null distribution of $\delta_2(n)$ by showing that it is equivalent to the distribution of the maximum of the absolute value of a simple random walk. The latter, however, can be derived by a rather straighforward application of the reflection principle to count lattice paths in the plane restricted by two parallel lines. The case $r > 2$ has been studied by Kiefer (1955, 1959), who has argued that for $r = 3$ distance criteria like (1) have good power properties for certain classes of alternatives. David (1958) discusses the case $r = 3$ and derives the null distribution by applying the reflection principle to two-dimensional lattice paths on a hexagonal grid restricted to stay inside an equilateral triangle centered at the origin. Takács (1996) studied also the case $r = 3$ by considering 3-dimensional paths with step set $\mathcal{S} = \{(1, 0, -1), (-1, 1, 0), (0, -1, 1)\}$ which start and terminate at the origin and are not allowed to touch boundaries represented by the planes $x = k, y = k$ and $z = k$. Note that due to linear dependence in the step set $\mathcal{S}$ paths are restricted to a triangular region, as was the case in David's approach. The solution is found again by an application of the reflection principle. Takács also carries out a thorough study of the asymptotic distribution of $\delta_3(n)$. It is quite interesting that earlier Filaseta (1985) had already solved this problem for general $r \geq 2$. Actually, he derived an interesting counting formula for a particular type of an r-candidate ballot problem and showed that one of its potential applications is just the derivation of the null distribution of (1).

In Section 2 we derive the null distribution of $\delta_r(n)$ by considering lattice paths in r-dimensional space with standard steps in the positive direction, i.e., steps are given by the unit vectors $\mathbf{e}_i, i = 1, 2, \ldots, r$. By a simple transformation we show that for some positive integer k the number of ways the event $\{n\delta_r(n) < k\}$ can occur is just the number of paths $\mathbf{X}$ with the property that for each point $\mathbf{X}_m$ on the path there holds the chain of inequalities

$$x_{1,m} > x_{2,m} > \ldots > x_{r,m} > x_{1,m} - rk.$$

Indeed, the enumeration of such paths is a well studied problem in combinatorics. Again the reflection principle comes into play as we have to count paths in alcoves of affine (and therefore infinite) Weyl groups; for references on the technical background of this topic see Gessel and Zeilberger (1992), Grabiner (2002) and Krattenthaler (2007). The resulting formula is essentially an r-fold summation of determinants of order $r \times r$, from a computational point of view a real challenge. Regarding asymptotics (using a standard diffusion scaling) at a first sight no simplification can be expected. Actually, Filaseta (1985) has also given an asymptotic formula, again a multiple summation of determinants. However, we are able to show that considerable simplifications are indeed possible by reducing these determinants to Vandermonde form. In particular, it turns out that the limit

$$\lim_{n\to\infty} P(\sqrt{n}\delta_r(n) < x) = H_r(x) \tag{2}$$

exists. $H_r(x)$ is the distribution function of a positive random variable and is given by the multivariate Dirichlet series:

$$H_r(x) = \sum_{v_1+\ldots+v_r=0} e^{-\frac{1}{2}r^2x^2\sum_{i=1}^r v_i^2} \prod_{1\leq i<j\leq r} (1 - e^{x^2(i-j+r(v_i-v_j))}). \tag{3}$$

The exact results of Section 2 only hold when all samples have equal size n. Unfortunately, in the case of unequal sample sizes and $r \geq 3$ samples no combinatorial results are currently available as the resulting lattice paths are no longer reflectable. Still we are able to prove in Section 4 that by a proper rescaling a test statistic based on a linear transformation of the circular differences $\delta_{ij}(n_i, n_j)$ again has the distribution (3).

In the appendix of this paper some tables are presented for the exact and the asymptotic distribution of $\delta_r(n)$.

2. Lattice path counting and the distribution of $\delta_r(n)$

Suppose we have $r \geq 2$ samples of equal size n drawn from the same continuous distribution. Let $\xi_1, \xi_2, \ldots, \xi_{rn}$ be independent random variables each having the same continuous distribution $F(x)$ and let ξ_i come from the r samples with the convention that the ξ_i are organized in samples $S_1, S_2, \ldots, S_r$ in the following way:

$$\begin{aligned} S_1 &= (\xi_1, \xi_2, \ldots, \xi_n) \\ S_2 &= (\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{2n}) \\ &\cdots \\ S_r &= (\xi_{(r-1)n+1}, \xi_{(r-1)n+2}, \ldots, \xi_{rn}) \end{aligned}$$

Furthermore let $F_{n,1}(x), F_{n,2}(x), \ldots, F_{n,r}(x)$ denote the empirical distribution functions of these samples, i.e., $F_{n,i}(x)$ equals the relative frequency of variables among $\xi_{(i-1)n+1}$, $\xi_{(i-1)n+2}$, $\ldots, \xi_{in}$ less than or equal to x.

Define the statistic

$$\delta_{i,j}(n) = \sup_x \left[F_{n,i}(x) - F_{n,j}(x)\right], \quad i, j = 1, \ldots, r. \tag{4}$$

The random variables $\delta_{i,j}(n), i, j = 1, 2, \ldots, r$, are distribution-free statistics, since $F(x)$ is a continuous distribution and therefore the joint distribution of random variables $\delta_{i,j}(n)$ does not depend on $F(x)$.

To test whether the samples $S_1, \ldots, S_r$ are all drawn from $F(x)$ we consider the statistic

$$\delta_r(n) = \max\left[\delta_{1,2}(n), \delta_{2,3}(n), \ldots, \delta_{r-1,r}(n), \delta_{r,1}(n)\right]. \tag{5}$$

Before embarking on the general case $r \geq 3$ by methods of lattice path counting it is quite instructive to review briefly the case $r = 2$ and its combinatorial aspects. In this situation the test statistic will be:

$$\delta_2(n) = \max\left[\delta_{1,2}(n), \delta_{2,1}(n)\right] = \sup |F_{n,1}(x) - F_{n,2}(x)|.$$

The exact distribution of $\delta_2(n)$ has been known for a long time, for a rather comprehensive treatment the reader is referred to Hajek and Sidak (1967) and Durbin (1973). A combinatorial derivation of the exact distribution of $\delta_2(n)$ was first given by Gnedenko and Korolyuk (1951). Let us briefly recall how Gnedenko's technique works.

Combine the samples S_1 and S_2 into a single sample in such a way that the $2n$ elements are arranged in nondecreasing order of magnitude. From this combined sample we construct a 2-dimensional lattice path as follows: start in the point $(k, 0)$, where k is a positive integer. For each element in the combined sample draw a horizontal step of unit size, if the element comes from S_1 and draw a vertical step of unit size, if it comes from S_2. As a result, we get a lattice path in the plane, leading from the point $(k, 0)$ to the point $(n+k, n)$. In total there are $\binom{2n}{n}$ such paths. In order to find $P(n\delta_2(n) < k)$ we have to consider only those lattice paths, for which in every point there holds

$$x_1 > x_2 > x_1 - 2k$$

In other words, we have to count the number of paths from $(k, 0)$ to $(n + k, k)$ which do not touch or cross the lines $x_2 = x_1$ and $x_2 = x_1 - 2k$ (see Figure 1). The number of such paths can be determined by means of the *reflection*

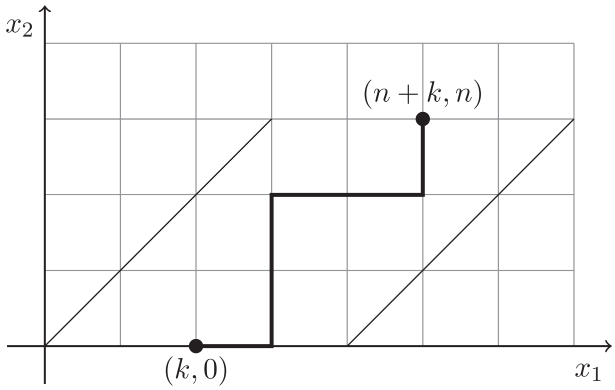

FIGURE 1.

principle. The starting point $(k, 0)$ is reflected repeatedly at the lines $x_2 = x_1$ and $x_2 = x_1 - 2k$ and using an inclusion-exclusion argument, see e.g. Mohanty (1979, pp. 6, 7), one finally finds that

$$\binom{2n}{n} P(n\delta_2(n) < k) = \sum_i \left[\binom{2n}{n+2ki} - \binom{2n}{n+k+2ki} \right]. \tag{6}$$

Interestingly, it turns out that essentially the same argument which led to (6) can be used for the general case $r > 2$. To see this let us first combine the samples $S_1, \dots, S_r$ into a single sample in such a way that the rn elements are arranged in nondecreasing order of magnitude. Next we construct a lattice path $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_{rn})$ in r-dimensional space with

$$\mathbf{Y}_m = (y_{1,m}, y_{2,m}, \dots, y_{r,m}).$$

The path starts at the origin and terminates after rn steps in the lattice point $(n, n, \dots, n)$, where the steps are the r unit vectors $\mathbf{e}_i$, $i = 1, \dots, r$. In particular, if the j-th element in the combined sample comes from sample S_ℓ, then the j-the step $\mathbf{Y}_j - \mathbf{Y}_{j-1}$ is given by $\mathbf{e}_\ell$.

Consider now the event

$$\{n\delta_{1,2}(n) < k_1, n\delta_{2,3}(n) < k_2, \dots, n\delta_{r-1,r}(n) < k_{r-1}, n\delta_{r,1}(n) < k_r\}.$$

for positive integers $k_i, i = 1, 2, \ldots, r$. By definition (4) this event is equivalent to the event that in each point $\mathbf{Y}_m$ of the path $\mathbf{Y}$ there holds:

$$\begin{aligned} y_{1,m} - y_{2,m} < k_1, \quad y_{2,m} - y_{3,m} < k_2, \ldots, \\ y_{r-1,m} - y_{r,m} < k_{r-1}, \quad y_{r,m} - y_{1,m} < k_r. \end{aligned} \tag{7}$$

Let us put for $i = 0, 1, \ldots, r-1$:

$$\beta_i = \sum_{\ell=1}^{r-i} k_\ell, \quad \text{and} \quad \beta_r = 0, \tag{8}$$

then the chain of inequalities (7) can also be written as

$$\begin{aligned} y_{1,m} &< \beta_{r-1} + y_{2,m}, \\ \beta_{r-1} + y_{2,m} &< \beta_{r-2} + y_{3,m} \\ \beta_{r-2} + y_{3,m} &< \beta_{r-3} + y_{4,m} \\ &\ldots \\ \beta_1 + y_{r,m} &< \beta_0 + y_{r,m}. \end{aligned}$$

From the paths $\mathbf{Y}$ just described we construct new paths $\mathbf{X}$ by simply relabeling and shifting the coordinates in each point so that for each point $\mathbf{X}_m$ on $\mathbf{X}$ we have

$$\begin{aligned} x_{1,m} &= \beta_1 + y_{r,m} \\ x_{2,m} &= \beta_2 + y_{r-1,m} \\ &\ldots \\ x_{r,m} &= \beta_r + y_{1,m} = y_{1,m}. \end{aligned}$$

Clearly, the set of all paths $\{\mathbf{Y}\}$ has the same cardinality as the set $\{\mathbf{X}\}$.

Now any path $\mathbf{X}$ starts in a point $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r)$ and terminates after rn steps in a point $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ with

$$\eta_i = \beta_i, \quad \lambda_i = n + \beta_i, \qquad i = 1, 2, \ldots, r \tag{9}$$

and for each point $\mathbf{X}_m$ on $\mathbf{X}$ there holds

$$x_{1,m} > x_{2,m} > \ldots > x_{r,m} > x_{1,m} - \beta_0. \tag{10}$$

These inequalities define a scaled alcove $\mathcal{A}_r$ of the affine Weyl group A_{r-1}. The number $L_{\mathcal{A}_r}(\boldsymbol{\eta}, \boldsymbol{\lambda})$ of paths $\mathbf{X}$ going from $\boldsymbol{\eta}$ to $\boldsymbol{\lambda}$ and staying strictly in this alcove is well known and given by

Theorem 2.1.

$$L_{\mathcal{A}_r}(\boldsymbol{\eta}, \boldsymbol{\lambda}) = (rn)! \sum_{v_1+v_2+\ldots+v_r=0} \det\left[\frac{1}{(n+\beta_j-\beta_i+\beta_0 v_i)!}\right]_{r\times r}, \tag{11}$$

with the convention that $1/(-m)! = 0$ *for positive integers* m.

This formula is originally due to Filaseta (1985), but contained as a special case in the more general result of Gessel and Zeilberger (1992), an important reference is also Krattenthaler (1994). A thorough discussion of random walks in alcoves of affine Weyl groups can be found in Grabiner (2002).

Now let $L(\boldsymbol{\eta}, \boldsymbol{\lambda})$ be the number of paths from $\boldsymbol{\eta} \to \boldsymbol{\lambda}$ without any restriction. It is given by the multinomial coefficient

$$L(\boldsymbol{\eta}, \boldsymbol{\lambda}) = \frac{(\sum_{i=1}^{r}(\lambda_i - \eta_i))!}{\prod_{i=1}^{r}(\lambda_i - \eta_i)!} = \frac{(rn)!}{(n!)^r}. \tag{12}$$

Under the null hypothesis that all samples come from the same distribution $F(x)$ any path from $\boldsymbol{\eta} \to \boldsymbol{\lambda}$ has the same probability $1/L(\boldsymbol{\eta}, \boldsymbol{\lambda})$, hence we obtain for the joint distribution of the distances $\delta_{i,i+1}(n)$:

$$\begin{aligned} P(n\delta_{i,i+1}(n) < k_i, i = 1, \ldots, r-1, n\delta_{r,1} < k_r) = \\ = \sum_{v_1+\ldots+v_r=0} \det\left[\frac{n!}{(n + \beta_j - \beta_i + \beta_0 v_i)!}\right]_{r\times r}. \end{aligned} \tag{13}$$

A considerable simplification results from setting $k_r = +\infty$. Then also $\beta_0 = +\infty$ and the multiple summation in (13) only the term for $v_1 = v_2 = \ldots = v_r = 0$ can be different from zero. Thus we get for the marginal:

$$P(n\delta_{i,i+1}(n) < k_i, i = 1, \ldots, r-1) = \det\left[\frac{n!}{(n + \beta_j - \beta_i)!}\right]_{r\times r} \tag{14}$$

Note that (14) is the probability that the random walk $\mathbf{X}$ moves from $\boldsymbol{\eta}$ to $\boldsymbol{\lambda}$ and stays strictly in the Weyl chamber defined by

$$x_1 > x_2 > \cdots > x_r.$$

Thus (14) is an immediate consequence of the classical r-candidate ballot theorem, see Watanabe and Mohanty (1987) and Zeilberger (1983). An interesting reference is Karlin and McGregor (1959), where (14) is given essentially as a corollary of a more general result on noncoincidence probabilities of strong Markov processes.

In the sequel we will concentrate on the test statistic $\delta_r(n)$ defined in (5). For this purpose we put $k_i = k$, hence $\beta_i = k(r-i)$ for $k = 1, 2, \ldots, r$ and from (13) it follows:

Corollary 2.2.

$$P(n\delta_r(n) < k) = \sum_{v_1+\ldots+v_r=0} \det\left[\frac{n!}{(n + k(i-j) + v_i rk)!}\right]_{r\times r}. \tag{15}$$

Unfortunately, the numerical evaluation of (15) is by no means trivial and it poses some really challenging computational problems. We have prepared tables for various values of n and r, which can be found in the appendix of this paper.

There is a remarkable asymptotic formula for $L_{\mathcal{A}_r}(\boldsymbol{\eta}, \boldsymbol{\lambda})$ due to Krattenthaler (2007, Theorem 12). Using Krattenthaler's results we obtain the following approximation of $P(n\delta_r(n) < k)$ when k is fixed and $n \to \infty$:

$$P(n\delta_r(n) < k) \sim \frac{(n!)^r}{(rn)!} \frac{2^{r(r-1)}}{(rk)^{r-1}} \left(\frac{\sin \frac{\pi}{k}}{\sin \frac{\pi}{rk}} \right)^{rn} \prod_{1 \le i < j \le r} \sin^2 \left(\frac{\pi(j-i)}{r} \right). \tag{16}$$

This formula is amazing in many respects, notably because of its astounding simplicity compared to (15) and its accuracy even for small values of n. Just to get an impression, see Table 1. When studying the proof of Theorem 12 in

	$r=3, n=10$		$r=3, n=20$		$r=4, n=10$		$r=4, n=20$	
k	exact	approx.	exact	approx.	exact	approx.	exact	approx.
2	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.0764	0.0764	0.0009	0.0009	0.0302	0.0302	0.0001	0.0001
4	0.4201	0.4200	0.0502	0.0502	0.3118	0.3114	0.0168	0.0168
5	0.7524	0.7499	0.2499	0.2499	0.6839	0.6730	0.1534	0.1534
6			0.5214	0.5212			0.4175	0.4161
7			0.7423	0.7395			0.6716	0.6607

Table 1. Exact and approximate values for $k \le \lceil \sqrt{n} \rceil$

Krattenthaler's (2007) paper, we found that from his formula (3.2) there follows an interesting representation of (15) in terms of trigonometric functions. In particular:

$$\begin{aligned} P(n\delta_r(n) < k) = \frac{(n!)^r}{(rk)^r (nr)!} & \sum_{v_1, v_2, \ldots v_r = 0}^{rk-1} \left[\sum_{i=1}^{r} \omega^{v_i} \right]^{nr} \omega^{-n \sum_{i=1}^r v_i} \\ & \times \prod_{1 \le i < j \le r} \left[1 - \omega^{k(v_i - v_j)} \right]. \end{aligned} \tag{17}$$

where $\omega = e^{2\pi i/rk}$. The nice thing about (17) is that the determinant disappeared! Moreover, the Vandermondian product is actually independent of k, since k cancells. Note also that in the summation above only those indices $v_1, \ldots, v_r$ have to be considered where the v_i are pairwise different and $v_i - v_j$ is not divisible by r.

3. The asymptotic distribution of $\delta_r(n)$

In this section we will prove the following theorem:

Theorem 3.1. *The limit*

$$\lim_{n \to \infty} P(\sqrt{n}\delta_r(n) < x) = H_r(x) \tag{18}$$

exists and $H_r(x)$ is the distribution function of a positive random variable and given by

$$H_r(x) = \sum_{v_1 + \ldots + v_r = 0} e^{-\frac{1}{2} r^2 x^2 \sum_{i=1}^r v_i^2} \prod_{1 \le i < j \le r} (1 - e^{x^2(i - j + r(v_i - v_j))}). \tag{19}$$

Proof. Put $k = \lfloor x\sqrt{n} \rfloor$, so that $k = x\sqrt{n} - \epsilon$ for some $0 \le \epsilon < 1$. Next, let us expand the determinant in (15):

$$P(n\delta_r(n) < k) = \sum_{v_1+\ldots+v_r=0} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma \frac{(n!)^r}{\prod_{i=1}^r (n + k(i - \sigma_i) + krv_i)!}.$$

Here $\sigma = (\sigma_1, \ldots, \sigma_r)$ is a permutation, $(-1)^\sigma$ its sign and $\mathbb{S}_r$ denotes the symmetric group of order r. Let f_n denote the generic term in this expansion, i.e.,

$$f_n = \frac{(n!)^r}{\prod_{i=1}^r (n + k(i - \sigma_i) + krv_i)!}.$$

Put $\alpha_i = i - \sigma_i + rv_i$. Thus

$$f_n = \frac{(n!)^r}{\prod_{i=1}^r (n + \alpha_i \lfloor x\sqrt{n} \rfloor)!}.$$

In what follows it will be helpful to note that:

$$\sum_{i=1}^r \alpha_i = \sum_{i=1}^r (i - \sigma_i) + r \sum_{i=1}^r v_i = 0.$$

We approximate the factorials in the numerator and denominator of $f_n = a_n/b_n$ by means of Stirling's formula,

$$n! = n^{n+1/2} e^{-n} \sqrt{2\pi} \left(1 + \frac{1}{12n} + O(n^{-2})\right).$$

For the numerator we have

$$a_n = n^{rn+r/2} e^{-rn} (2\pi)^{r/2} \left(1 + \frac{r}{12n} + O(n^{-2})\right).$$

By the same token, one obtains for the denominator

$$\begin{aligned} b_n &= \prod_{i=1}^r (n + \alpha_i \lfloor x\sqrt{n} \rfloor)^{n + \lfloor x\sqrt{n} \rfloor + 1/2} e^{-n - \alpha_i \lfloor x\sqrt{n} \rfloor} (2\pi)^{1/2} \left(1 + \frac{1}{12n} + O(n^{-2})\right) \\ &= \left[e^{-rn} (2\pi)^{r/2} \prod_{i=1}^r (n + \alpha_i \lfloor x\sqrt{n} \rfloor)^{n + \lfloor x\sqrt{n} \rfloor + 1/2} \right] \left(1 + \frac{r}{12n} + O(n^{-2})\right). \end{aligned}$$

Consider now the log of the i-th factor in the product above:

$$\left(n + \alpha_i \lfloor x\sqrt{n} \rfloor + \frac{1}{2}\right) \ln n + \left(n + \alpha_i \lfloor x\sqrt{n} \rfloor + \frac{1}{2}\right) \ln \left(1 + \frac{\alpha_i \lfloor x\sqrt{n} \rfloor}{n}\right).$$

By Taylor's Theorem the second log may be expanded to

$$\ln \left(1 + \frac{\alpha_i \lfloor x\sqrt{n} \rfloor}{n}\right) = \frac{\alpha_i \lfloor x\sqrt{n} \rfloor}{n} - \frac{\alpha_i^2 (\lfloor x\sqrt{n} \rfloor)^2}{2n^2} + O(n^{-3/2}).$$

It follows that

$$\begin{aligned} \left(n + \alpha_i \lfloor x\sqrt{n} \rfloor + \frac{1}{2}\right) \ln \left(1 + \frac{\alpha_i \lfloor x\sqrt{n} \rfloor}{n}\right) &= \alpha_i \lfloor x\sqrt{n} \rfloor + \frac{\alpha_i^2 (\lfloor x\sqrt{n} \rfloor)^2}{2n} + O(n^{-1/2}) \\ &= \alpha_i \lfloor x\sqrt{n} \rfloor + \frac{\alpha_i^2 x^2}{2} + O(n^{-1/2}). \end{aligned}$$

Therefore the product in b_n is approximated by

$$\prod_{i=1}^{r} n^{n+\alpha_i\lfloor x\sqrt{n}\rfloor+1/2}\cdot e^{\alpha_i\lfloor x\sqrt{n}\rfloor+\alpha_i^2x^2/2}\Big(1+O(n^{-1/2})\Big)$$
$$= n^{rn+r/2}\prod_{i=1}^{r} e^{\alpha_i^2x^2/2}\Big(1+O(n^{-1/2})\Big),$$

because $\sum_{i=1}^{r}\alpha_i=0$. Hence

$$b_n=\left[e^{-rn}(2\pi)^{r/2}n^{rn+r/2}\prod_{i=1}^{r}e^{\alpha_i^2x^2/2}\right]\Big(1+O(n^{-1/2})\Big).$$

This implies that for large n:

$$f_n=\prod_{i=1}^{r}e^{-\alpha_i^2x^2/2}\Big(1+O(n^{-1/2}\Big)\to\prod_{i=1}^{r}e^{-x^2(i-\sigma_i+rv_i)^2/2}.$$

Returning to (15), we find that

$$\lim_{n\to\infty}P(\sqrt{n}\delta_r(n)<x)=\lim_{n\to\infty}\frac{L_{\mathcal{A}_r}(\boldsymbol{\eta},\boldsymbol{\lambda})}{L(\boldsymbol{\eta},\boldsymbol{\lambda})}$$
$$=\sum_{v_1+\ldots+v_r=0}\sum_{\sigma\in\mathbb{S}_r}(-1)^{\sigma}\prod_{i=1}^{r}e^{-x^2(i-\sigma_i+rv_i)^2/2}$$
$$=\sum_{v_1+\ldots+v_r=0}\det_{1\le i,j\le r}\left[e^{-x^2(i-j+rv_i)^2/2}\right]. \tag{20}$$

Let us now have a closer look at the determinants occuring here. They can be written as:

(21)

$$\det\left[e^{-x^2(i-j+rv_i)^2/2}\right]=\det\left[e^{-x^2[i^2-2ij+j^2+2irv_i-2jrv_i+r^2v_i^2]/2}\right]$$
$$=e^{-x^2[\sum_{i=1}^r i^2+r\sum_{i=1}^r iv_i+\frac{1}{2}r^2\sum_{i=1}^r v_i^2]}\cdot\det\left[e^{x^2(i+rv_i)j}\right]$$
$$=e^{-x^2[r(r+1)(2r+1)/6+r\sum_{i=1}^r iv_i+\frac{1}{2}r^2\sum_{i=1}^r v_i^2]}\cdot\det\left[e^{x^2(i+rv_i)j}\right].$$

Observe that the determinants on the right hand side above are of Vandermonde type. Indeed, putting $z_i=e^{x^2(i+rv_i)}$, we obtain

$$\det_{1\le i,j\le r}\left[e^{x^2(i+rv_i)j}\right]=\det_{1\le i,j\le r}\left[z_i^j\right]=\prod_{i=1}^{r}z_i\cdot\det_{1\le i,j\le r}\left[z_i^{j-1}\right]$$
$$=e^{x^2r(r+1)/2}\prod_{1\le i,j\le r}(z_j-z_i).$$

Thus

$$H_r(x)=$$
$$=e^{-\frac{1}{3}x^2(r^3-r)}\sum_{v_1+\ldots+v_r=0}e^{-rx^2(\sum_i iv_i+\frac{1}{2}r\sum_i v_i^2)}\prod_{1\le i<j\le r}\left[e^{x^2(j+rv_j)}-e^{x^2(i+rv_i)}\right].$$

The product in $H_r(x)$ may be simplified further:

$$\prod_{1\le i<j\le r}\left[e^{x^2(j+rv_j)}-e^{x^2(i+rv_i)}\right]=\prod_{1\le i<j\le r}e^{x^2(j+rv_j)}\left[1-e^{x^2(i-j+r(v_i-v_j))}\right]$$
$$=e^{x^2[\sum_{j=2}^{r}j(j-1)+r\sum_{j=2}^{r}(j-1)v_j]}\prod_{1\le i<j\le r}\left[1-e^{x^2(i-j+r(v_i-v_j))}\right]$$
$$=e^{x^2[\frac{1}{3}r(r^2-1)+r\sum_{j=1}^{r}jv_j]}\prod_{1\le i<j\le r}\left[1-e^{x^2(i-j+r(v_i-v_j))}\right],$$

and this finally proves (19). □

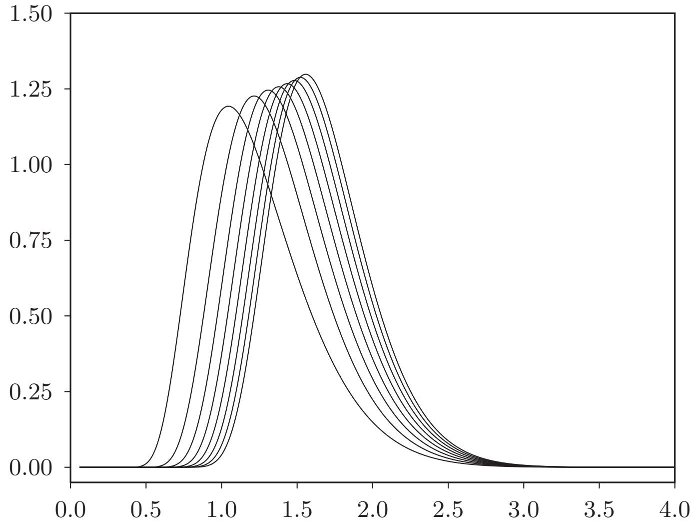

FIGURE 2. The density $H_r'(x)$ for $r = 2, 3, \ldots, 9$ (from left to right)

4. Unequal Sample Sizes

The purpose of this section is to shed more light on the case where the sample sizes are not necessarily identical. Now the situation is much more complicated even when $r = 2$, since the reflection principle is no longer applicable. For this case combinatorial results are still available again in terms of determinants, but this time the size of the determinants explicitly depends on the sample size. It can be shown that for $r = 2$ one has to count the number of lattice paths restricted by two general boundaries, a problem which can be reduced to counting families of nonintersecting paths in the plane, see the classical paper of Kreweras (1965)[1], Steck (1969) and Mohanty (1979, pp. 33 and especially pp. 101). A treatment based on Markov chains may be found in the booklet of Durbin (1973, pp. 39).

[1]An english translation kindly has been made available to us by Prof. Mohanty.

In the case $r > 2$ the situation becomes truly complicated and results for finite sample sizes are still unknown. There is some hope that the exact distribution of the test statistic can be found using a technique for counting higher dimensional paths due to Handa and Mohanty (1976), but this is subject to further research.

However, an asymptotic result can be established.

Write $\mathbf{N} = (n_1, \ldots, n_r)$ and $\mathbf{F_N}$ for the process on the space $\mathcal{D} = \mathcal{D}([0,1])^r$ such that $[\mathbf{F_N}(t)]_i = F_{n_i,i}(t)$.

Assume without loss of generality that the samples are drawn from the uniform distribution on $[0, 1]$.

Consider the process $\mathbf{E_N}$ on $\mathcal{D}$ for which $[\mathbf{E_N}(t)]_i = E_{n_i,i} = \sqrt{n_i}(F_{n_i,i}(t) - t)$. If all n_i tend to infinity, $\mathbf{E_N} \Rightarrow \mathbf{B}$ weakly on $\mathcal{D}$, where the elements of $\mathbf{B} = (B_1, \ldots, B_r)$ are independent Brownian bridges.

Write Φ for the functional $\Phi(\mathbf{f}) = \max_i \sup_{0\leq t\leq 1} f_i(t)$ on $\mathcal{D}$, and for $\mathbf{x} = (x_1, \ldots, x_r)$, let $\Delta(\mathbf{x}) = (x_1 - x_2, \ldots, x_r - x_1)$ denote the "circular" difference of the elements of $\mathbf{x}$. For the case where all sample sizes are equal, the test statistic considered in the previous section is:

$$\delta_r(n) = \Phi(\Delta(\mathbf{F}_{n,\ldots,n})). \tag{22}$$

Clearly, $\sqrt{n}\Delta(\mathbf{F}_{n,\ldots,n}) = \Delta(\mathbf{E}_{n,\ldots,n})$, so that $\sqrt{n}\delta_r(n) = \Phi(\Delta(\mathbf{E}_{n,\ldots,n}))$. By the Continuous Mapping Theorem, this converges to $\Phi(\Delta(\mathbf{B}))$ in distribution, and (19) provides a formula for the corresponding distribution function $H_r(x)$.

For the general case where the sample sizes are not necessarily equal, we use the test statistic

$$\Phi(\mathbf{A_N}\Delta(\mathbf{F_N})), \tag{23}$$

where $\mathbf{A_N}$ is a suitable (not necessarily diagonal) rescaling matrix chosen in a way that $\mathbf{A_N}\Delta(\mathbf{F_N}) \Rightarrow \Delta(\mathbf{B})$ so we get the same explicit formula for the limit distribution of $\Phi(\mathbf{A_N}\Delta(\mathbf{F_N}))$ as in the case of equal sample sizes.

Let $\mathbf{T} = \mathbf{T}_\Delta$ be the $r \times r$ matrix representing the linear transformation Δ (such that $\Delta(\mathbf{x}) = \mathbf{Tx}$). Clearly, the rank of T is $r - 1 = k$. Let $\mathbf{T} = \mathbf{UDV}'$ be the reduced singular value decomposition of $\mathbf{T}$ where all matrices have full rank, and the columns of $\mathbf{U}$ and $\mathbf{V}$ are orthonormal.

Let $\mathbf{\Lambda_N} = \operatorname{diag}(1/n_1, \ldots, 1/n_r)$. Then $\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V}$ is symmetric and invertible, and hence its symmetric inverse square root $(\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V})^{-1/2}$ is well-defined.

Theorem 4.1. *Let*

$$\mathbf{A_N} = \mathbf{UD}(\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V})^{-1/2}\mathbf{D}^{-1}\mathbf{U}'. \tag{24}$$

Suppose also that the components of $\mathbf{N}$ *tend to infinity in such a way that* $\lim_{\mathbf{N}}(\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V})^{-1/2}\mathbf{V}'\mathbf{\Lambda_N}^{1/2}$ *exists. Then* $\mathbf{A_N}\Delta(\mathbf{F_N}) \Rightarrow \Delta(\mathbf{B})$.

Proof. For the proof, write $\mathbf{C_N} = \mathbf{UD}(\mathbf{V}'\mathbf{\Lambda_N V})^{-1/2}\mathbf{V}'\mathbf{\Lambda_N^{1/2}}$ and note that

$$\begin{aligned}\mathbf{C_N C_N'} &= \mathbf{UD}(\mathbf{V}'\mathbf{\Lambda_N V})^{-1/2}\mathbf{V}'\mathbf{\Lambda_N^{1/2}}\mathbf{\Lambda_N^{1/2}}\mathbf{V}(\mathbf{V}'\mathbf{\Lambda_N V})^{-1/2}\mathbf{DU}' \\ &= \mathbf{UD}^2\mathbf{U}' \\ &= \mathbf{TT}'.\end{aligned}$$

As

$$F_{n_i,i}(t) - F_{n_j,j}(t) = E_{n_i,i}(t)/\sqrt{n_i} - E_{n_j,j}(t)/\sqrt{n_j}, \tag{25}$$

we have

$$\Delta(\mathbf{F_N}) = \Delta(\mathbf{\Lambda_N^{1/2}E_N}) = \mathbf{T\Lambda_N^{1/2}E_N} = \mathbf{UDV}'\mathbf{\Lambda_N^{1/2}E_N}, \tag{26}$$

and hence

$$\mathbf{A_N}\Delta(\mathbf{F_N}) = \mathbf{UD}(\mathbf{V}'\mathbf{\Lambda_N V})^{-1/2}\mathbf{D}^{-1}\mathbf{U}'\mathbf{UDV}'\mathbf{\Lambda_N^{1/2}E_N} = \mathbf{C_N E_N}. \tag{27}$$

The assumptions imply that $\mathbf{C_N}$ has a limit, say $\mathbf{C}$. Thus, $\mathbf{A_N}\Delta(\mathbf{F_N}) = \mathbf{C_N E_N} \Rightarrow \mathbf{CB}$. We complete the proof by showing that this limit has the same covariance function as $\Delta(\mathbf{B})$.

Now, $\operatorname{cov}(\mathbf{B}(s), \mathbf{B}(t)) = (\min(s,t) - st)\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Thus,

$$\operatorname{cov}(\Delta(\mathbf{B}(s)), \Delta(\mathbf{B}(t))) = (\min(s,t) - st)\mathbf{TT}'. \tag{28}$$

On the other hand,

$$\begin{aligned}\operatorname{cov}(\mathbf{C_N E_N}(s), \mathbf{C_N E_N}(t)) &= \mathbf{C_N} \operatorname{cov}(\mathbf{E_N}(s), \mathbf{E_N}(t))\, \mathbf{C_N'} \\ &\approx \mathbf{C_N} \operatorname{cov}(\mathbf{B}(s), \mathbf{B}(t))\, \mathbf{C_N'} \\ &= (\min(s,t) - st)\mathbf{C_N C_N'} \\ &= (\min(s,t) - st)\mathbf{TT}',\end{aligned}$$

completing the proof. □

Corollary 4.2. *Under the above conditions,*

$$\lim_{\mathbf{N}} P(\Phi(\mathbf{A_N}\Delta(\mathbf{F_N})) < x) = H_r(x), \tag{29}$$

where $H_r(x)$ is given by (19) in Theorem 3.1.

Some remarks are in order:

Remark 1. Assuming that $\mathbf{C_N}$ has a limit may not be necessary: to show identity of the covariance functions, we really only need that the $\mathbf{C_N}$ sequence is uniformly bounded. This might also be enough for showing that $\mathbf{C_N E_N}$ converges weakly.

Remark 2. Let $\bar{n} = \operatorname{mean}(\mathbf{N})$ be the average sample size. If all limits $n_i/\bar{n}$ exist, the condition of the theorem is clearly satisfied. (Equivalently, we can formulate the result by taking $\mathbf{N} \approx (\gamma_1, \ldots, \gamma_r)n$ for some positive reals γ_i and letting $n \to \infty$.)

Remark 3. If $r = 2$,

$$\mathbf{T} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2\mathbf{u}\mathbf{u}', \qquad \mathbf{u}' = (1, -1)/\sqrt{2}.$$

so that $\mathbf{U} = \mathbf{V} = \mathbf{u}$ and $\mathbf{D} = 2$, $\mathbf{T}\mathbf{T}' = \mathbf{U}\mathbf{D}^2\mathbf{U}' = 4\mathbf{u}\mathbf{u}' = 2\mathbf{T}$ and

$$\begin{aligned}\mathbf{A_N} &= \mathbf{U}\mathbf{D}(\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V})^{-1/2}\mathbf{D}^{-1}\mathbf{U}' \\ &= \mathbf{u}(\mathbf{u}'\mathbf{\Lambda_N}\mathbf{u})^{-1/2}\mathbf{u}' \\ &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \mathbf{u}\mathbf{u}'.\end{aligned}$$

Noting that

$$\begin{aligned}\mathbf{A_N}\Delta(\mathbf{F_N}) &= \mathbf{A_N}\mathbf{T}\mathbf{F_N} \\ &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \mathbf{u}\mathbf{u}'\mathbf{T}\mathbf{F_N} \\ &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \mathbf{T}\mathbf{F_N} \\ &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \Delta(\mathbf{F_N}),\end{aligned}$$

we see that we obtain the usual normalization for the two-sample Kolmogorov-Smirnow result.

Remark 4. If all sample sizes are equal, $\mathbf{\Lambda_N} = (1/n)\,\mathbf{I}$ and thus

$$\mathbf{A_N} = \mathbf{U}\mathbf{D}(\mathbf{V}'\mathbf{\Lambda_N}\mathbf{V})^{-1/2}\mathbf{D}^{-1}\mathbf{U}' = \sqrt{n}\mathbf{U}\mathbf{U}'$$

so that $\mathbf{A_N}\Delta(\mathbf{F_N}) = \mathbf{U}\mathbf{U}'\mathbf{U}\mathbf{D}\mathbf{V}'\mathbf{F_N} = \sqrt{n}\Delta(\mathbf{F_N})$ reduces to the standard scaling.

Remark 5. We can also use the above result to obtain joint distributions of componentwise maxima. Let $\Psi : \mathcal{D} \to \mathbb{R}^{r-1}$ be given by

$$\Psi(\mathbf{f}(t)) = (\sup_{0\le t\le 1} f_1(t), \ldots, \sup_{0\le t\le 1} f_{r-1}(t)). \tag{30}$$

Then under the conditions of the theorem,

$$\Psi(\mathbf{A_N}\Delta(\mathbf{F_N})) \to^d \Psi(\Delta(\mathbf{B})). \tag{31}$$

For $r = 3$, an explicit formula for the limiting distribution function can be obtained from Equation 66 in Takács (1994). (In fact, the result was first obtained by Ozols (1955).) Written out explicitly: let $(\xi_\mathbf{N}, \eta_\mathbf{N})$ be the 2-dimensional random vector $\Psi(\mathbf{A_N}\Delta(\mathbf{F_N}))$. Then under the conditions of the theorem,

$$P(\xi_\mathbf{N} < x, \eta_\mathbf{N} < y) \to 1 - e^{-x^2} - e^{-y^2} + 2e^{-x^2+2xy+y^2}$$

for $x, y > 0$.

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6. Tables

6.1. **Exact values.** Case $r = 3$:

k/n	5	10	20	50	100
2	0.0325	0.0001	0.0000	0.0000	0.0000
3	0.4968	0.0764	0.0009	0.0000	0.0000
4	0.8818	0.4201	0.0502	0.0000	0.0000
5	0.9881	0.7524	0.2499	0.0030	0.0000
6	1.0000	0.9216	0.5214	0.0322	0.0001
7		0.9815	0.7423	0.1227	0.0027
8		0.9969	0.8789	0.2720	0.0173
9		0.9997	0.9497	0.4446	0.0584
10		1.0000	0.9816	0.6056	0.1336
11			0.9940	0.7362	0.2375
12			0.9983	0.8325	0.3573
13			0.9996	0.8986	0.4790
14			0.9999	0.9413	0.5918
15			1.0000	0.9674	0.6897
16				0.9827	0.7704
17				0.9912	0.8342
18				0.9957	0.8830
19				0.9980	0.9193
20				0.9991	0.9455
21				0.9996	0.9639
22				0.9999	0.9766
23				0.9999	0.9852
24				1.0000	0.9908
25					0.9944
26					0.9967
27					0.9981
28					0.9989
29					0.9994
30					0.9997
31					0.9998
32					0.9999
33					1.0000

Case $r = 4$:

k/n	5	10	20	50	100
2	0.0094	0.0000	0.0000	0.0000	0.0000
3	0.3916	0.0302	0.0001	0.0000	0.0000
4	0.8455	0.3118	0.0168	0.0000	0.0000
5	0.9842	0.6839	0.1534	0.0003	0.0000
6	1.0000	0.8968	0.4175	0.0091	0.0000
7		0.9754	0.6716	0.0577	0.0003
8		0.9959	0.8419	0.1721	0.0038
9		0.9996	0.9335	0.3363	0.0207
10		1.0000	0.9755	0.5108	0.0648
11			0.9921	0.6642	0.1429
12			0.9978	0.7830	0.2498
13			0.9995	0.8671	0.3720
14			0.9999	0.9225	0.4952
15			1.0000	0.9568	0.6085
16				0.9770	0.7058
17				0.9883	0.7852
18				0.9943	0.8471
19				0.9974	0.8938
20				0.9988	0.9279
21				0.9995	0.9522
22				0.9998	0.9690
23				0.9999	0.9803
24				1.0000	0.9878
25					0.9925
26					0.9956
27					0.9974
28					0.9985
29					0.9992
30					0.9996
31					0.9998
32					0.9999
33					0.9999
34					1.0000

Case $r = 5$:

k/n	5	10	20	50	100
2	0.0028	0.0000	0.0000	0.0000	0.0000
3	0.3096	0.0123	0.0000	0.0000	0.0000
4	0.8108	0.2326	0.0058	0.0000	0.0000
5	0.9802	0.6219	0.0954	0.0000	0.0000
6	1.0000	0.8727	0.3354	0.0027	0.0000
7		0.9693	0.6080	0.0277	0.0000
8		0.9949	0.8064	0.1102	0.0009
9		0.9995	0.9176	0.2557	0.0076
10		1.0000	0.9694	0.4317	0.0321
11			0.9901	0.5996	0.0872
12			0.9972	0.7366	0.1761
13			0.9993	0.8367	0.2902
14			0.9999	0.9040	0.4152
15			1.0018	0.9463	0.5374
16				0.9714	0.6469
17				0.9854	0.7391
18				0.9929	0.8127
19				0.9967	0.8691
20				0.9985	0.9108
21				0.9994	0.9406
22				0.9998	0.9614
23				0.9999	0.9754
24				1.0000	0.9847
25					0.9907
26					0.9945
27					0.9968
28					0.9982
29					0.9990
30					0.9994
31					0.9997
32					0.9998
33					0.9999
34					1.0000

Case $r = 6$:

k/n	5	10	20	50	100
2	0.0009	0.0000	0.0000	0.0000	0.0000
3	0.2448	0.0050	0.0000	0.0000	0.0000
4	0.7775	0.1737	0.0021	0.0000	0.0000
5	0.9763	0.5656	0.0595	0.0000	0.0000
6	1.0000	0.8492	0.2695	0.0008	0.0000
7		0.9633	0.5504	0.0135	0.0000
8		0.9938	0.7724	0.0708	0.0002
9		0.9994	0.9020	0.1946	0.0028
10		1.0000	0.9635	0.3649	0.0161
11			0.9881	0.5413	0.0534
12			0.9967	0.6929	0.1243
13			0.9992	0.8074	0.2265
14			0.9998	0.8860	0.3483
15			1.0000	0.9360	0.4746
16				0.9658	0.5930
17				0.9826	0.6957
18				0.9915	0.7797
19				0.9961	0.8450
20				0.9983	0.8939
21				0.9993	0.9292
22				0.9997	0.9538
23				0.9999	0.9706
24				1.0000	0.9817
25					0.9888
26					0.9934
27					0.9961
28					0.9978
29					0.9988
30					0.9993
31					0.9996
32					0.9998
33					0.9999
34					1.0000

6.2. **The function $H_r(x)$ for various values of r and x.**

Case $r = 3$:

x	$H_3(x)$	x	$H_3(x)$	x	$H_3(x)$	x	$H_3(x)$
0.05	0.000000	1.05	0.184692	2.05	0.955146	3.05	0.999726
0.10	0.000000	1.10	0.239640	2.10	0.963545	3.10	0.999799
0.15	0.000000	1.15	0.298463	2.15	0.970521	3.15	0.999853
0.20	0.000000	1.20	0.359325	2.20	0.976282	3.20	0.999893
0.25	0.000000	1.25	0.420531	2.25	0.981012	3.25	0.999922
0.30	0.000000	1.30	0.480616	2.30	0.984875	3.30	0.999944
0.35	0.000000	1.35	0.538403	2.35	0.988013	3.35	0.999960
0.40	0.000000	1.40	0.593006	2.40	0.990547	3.40	0.999971
0.45	0.000000	1.45	0.643813	2.45	0.992582	3.45	0.999980
0.50	0.000001	1.50	0.690457	2.50	0.994209	3.50	0.999986
0.55	0.000018	1.55	0.732770	2.55	0.995501	3.55	0.999990
0.60	0.000154	1.60	0.770750	2.60	0.996522	3.60	0.999993
0.65	0.000798	1.65	0.804516	2.65	0.997325	3.65	0.999995
0.70	0.002876	1.70	0.834273	2.70	0.997953	3.70	0.999997
0.75	0.007941	1.75	0.860288	2.75	0.998441	3.75	0.999998
0.80	0.017945	1.80	0.882862	2.80	0.998819	3.80	0.999998
0.85	0.034768	1.85	0.902313	2.85	0.999110	3.85	0.999999
0.90	0.059754	1.90	0.918962	2.90	0.999332	3.90	0.999999
0.95	0.093424	1.95	0.933121	2.95	0.999501	3.95	0.999999
1.00	0.135429	2.00	0.945090	3.00	0.999630	4.00	1.000000

Case $r = 4$:

x	$H_4(x)$	x	$H_4(x)$	x	$H_4(x)$	x	$H_4(x)$
0.05	0.000000	1.05	0.101136	2.05	0.940642	3.05	0.999635
0.10	0.000000	1.10	0.144624	2.10	0.951689	3.10	0.999732
0.15	0.000000	1.15	0.195338	2.15	0.960888	3.15	0.999804
0.20	0.000000	1.20	0.251672	2.20	0.968501	3.20	0.999857
0.25	0.000000	1.25	0.311767	2.25	0.974763	3.25	0.999897
0.30	0.000000	1.30	0.373734	2.30	0.979885	3.30	0.999925
0.35	0.000000	1.35	0.435826	2.35	0.984049	3.35	0.999947
0.40	0.000000	1.40	0.496544	2.40	0.987416	3.40	0.999962
0.45	0.000000	1.45	0.554692	2.45	0.990122	3.45	0.999973
0.50	0.000000	1.50	0.609382	2.50	0.992286	3.50	0.999981
0.55	0.000000	1.55	0.660019	2.55	0.994006	3.55	0.999987
0.60	0.000005	1.60	0.706263	2.60	0.995366	3.60	0.999991
0.65	0.000052	1.65	0.747979	2.65	0.996435	3.65	0.999993
0.70	0.000313	1.70	0.785203	2.70	0.997272	3.70	0.999995
0.75	0.001290	1.75	0.818090	2.75	0.997922	3.75	0.999997
0.80	0.004010	1.80	0.846882	2.80	0.998426	3.80	0.999998
0.85	0.010050	1.85	0.871880	2.85	0.998813	3.85	0.999999
0.90	0.021293	1.90	0.893412	2.90	0.999110	3.90	0.999999
0.95	0.039516	1.95	0.911824	2.95	0.999335	3.95	0.999999
1.00	0.065974	2.00	0.927457	3.00	0.999506	4.00	1.000000

Case $r = 5$:

x	$H_5(x)$	x	$H_5(x)$	x	$H_5(x)$	x	$H_5(x)$
0.05	0.000000	1.05	0.056386	2.05	0.926361	3.05	0.999544
0.10	0.000000	1.10	0.088508	2.10	0.939980	3.10	0.999665
0.15	0.000000	1.15	0.129214	2.15	0.951351	3.15	0.999755
0.20	0.000000	1.20	0.177685	2.20	0.960782	3.20	0.999821
0.25	0.000000	1.25	0.232495	2.25	0.968554	3.25	0.999871
0.30	0.000000	1.30	0.291854	2.30	0.974920	3.30	0.999907
0.35	0.000000	1.35	0.353850	2.35	0.980101	3.35	0.999933
0.40	0.000000	1.40	0.416637	2.40	0.984294	3.40	0.999952
0.45	0.000000	1.45	0.478581	2.45	0.987668	3.45	0.999966
0.50	0.000000	1.50	0.538332	2.50	0.990366	3.50	0.999976
0.55	0.000000	1.55	0.594856	2.55	0.992513	3.55	0.999983
0.60	0.000000	1.60	0.647425	2.60	0.994211	3.60	0.999988
0.65	0.000004	1.65	0.695589	2.65	0.995546	3.65	0.999992
0.70	0.000037	1.70	0.739133	2.70	0.996591	3.70	0.999994
0.75	0.000224	1.75	0.778034	2.75	0.997403	3.75	0.999996
0.80	0.000945	1.80	0.812414	2.80	0.998032	3.80	0.999997
0.85	0.003034	1.85	0.842500	2.85	0.998517	3.85	0.999998
0.90	0.007861	1.90	0.868590	2.90	0.998887	3.90	0.999999
0.95	0.017200	1.95	0.891023	2.95	0.999169	3.95	0.999999
1.00	0.032882	2.00	0.910159	3.00	0.999383	4.00	0.999999

Case $r = 6$:

x	$H_6(x)$	x	$H_6(x)$	x	$H_6(x)$	x	$H_6(x)$
0.05	0.000000	1.05	0.031614	2.05	0.912297	3.05	0.999453
0.10	0.000000	1.10	0.054382	2.10	0.928416	3.10	0.999598
0.15	0.000000	1.15	0.085709	2.15	0.941909	3.15	0.999706
0.20	0.000000	1.20	0.125680	2.20	0.953125	3.20	0.999786
0.25	0.000000	1.25	0.173589	2.25	0.962385	3.25	0.999845
0.30	0.000000	1.30	0.228089	2.30	0.969979	3.30	0.999888
0.35	0.000000	1.35	0.287430	2.35	0.976169	3.35	0.999920
0.40	0.000000	1.40	0.349690	2.40	0.981183	3.40	0.999943
0.45	0.000000	1.45	0.412985	2.45	0.985220	3.45	0.999959
0.50	0.000000	1.50	0.475614	2.50	0.988451	3.50	0.999971
0.55	0.000000	1.55	0.536156	2.55	0.991022	3.55	0.999980
0.60	0.000000	1.60	0.593507	2.60	0.993057	3.60	0.999986
0.65	0.000000	1.65	0.646878	2.65	0.994658	3.65	0.999990
0.70	0.000005	1.70	0.695772	2.70	0.995910	3.70	0.999993
0.75	0.000040	1.75	0.739943	2.75	0.996885	3.75	0.999995
0.80	0.000228	1.80	0.779351	2.80	0.997639	3.80	0.999997
0.85	0.000933	1.85	0.814112	2.85	0.998220	3.85	0.999998
0.90	0.002942	1.90	0.844457	2.90	0.998665	3.90	0.999999
0.95	0.007564	1.95	0.870696	2.95	0.999003	3.95	0.999999
1.00	0.016515	2.00	0.893183	3.00	0.999260	4.00	0.999999