Non-algebraic convergence proofs for continuous-time fictitious play

Berger, Ulrich

Published in:
Dynamic Games and Applications

DOI:
10.1007/s13235-011-0033-4

Published: 01/01/2012

Document Version
Peer reviewed version

manuscript No.

Non-algebraic convergence proofs for continuous-time fictitious play

Ulrich Berger⋆

WU Vienna, Department of Economics
Augasse 2-6, 1090 Vienna, Austria
e-mail: ulrich.berger@wu.ac.at

August 4, 2011

Abstract In this technical note we use insights from the theory of projective geometry to provide novel and non-algebraic proofs of convergence of continuous-time fictitious play for a class of games. As a corollary we obtain a kind of equilibrium selection result, whereby continuous-time fictitious play converges to a particular equilibrium contained in a continuum of equivalent equilibria for symmetric 4×4 zero-sum games.

Journal of Economic Literature classification numbers: C72.

Key words Continuous-Time Fictitious Play, Best Response Dynamics, Learning, Projective Geometry.

1 Introduction

The discrete time fictitious play process is due to Brown (1951) and Robinson (1951). Originally this learning process, also called the "Brown-Robinson Learning Process", was proposed as an algorithm for calculating the value of a two-person zero-sum game. Since then it has become a standard learning process for boundedly rational players. In a fictitious play process each player believes that his opponent plays a stationary mixed strategy. He estimates this mixed strategy by the empirical distribution of pure strategies his opponent used during the history of the process and then plays a best response to this estimate.

⋆ I want to thank Josef Hofbauer, Ilias Kastanas, Andrew McLennan, Bill Sandholm, and two anonymous referees for valuable comments. Special thanks go to Francisco Franchetti for creating Figure 4 using the software Dynamo by Sandholm et al. (2011).

Robinson (1951) proved that under fictitious play the set of Nash equilibria is globally attractive for two-person zero-sum games, and Miyasawa (1961) proved convergence for 2×2-games. For games with more than two strategies per player convergence need not occur, however. Shapley (1964) demonstrated this with a 3×3 bimatrix game for which fictitious play ends up in an asymptotically stable limit cycle.

The cited results also hold for symmetric games, and for the continuous-time fictitious play process (Rosenmüller, 1971), henceforth CFP, or the best response dynamics (Matsui, 1992), which differs from this process only by a rescaling of time (see Hofbauer, 1995, Gaunersdorfer and Hofbauer, 1995, and Monderer and Shapley, 1996). For more recent results on the issue of convergence and nonconvergence of fictitious play and the best response dynamics see Berger (2005, 2007a, 2007b, 2008), Hofbauer and Sandholm (2009), Hahn (2010), and van Strien (2011).

CFP paths are typically the union of line segments in the state space. The point of this paper is to show how this geometric property can be exploited to provide new insights into the convergence properties of CFP paths, based on results from projective geometry. This is intended as a first starting point, and we want to propose this method as a useful tool for the further analysis of fictitious play.[1]

The remainder of this paper is structured as follows. Section 2 introduces fictitious play and the best response dynamics, and defines cyclic play. Section 3 reviews the principles of projective geometry and adds two results which establish the connection to cyclic CFP paths. In Section 4 we present the main result and add some examples of how to apply this result in well-known cases. A new result is derived from the main theorem in Section 5, where we show that CFP 'selects' a particular equilibrium from a continuum of equivalent equilibria in symmetric 4×4 zero-sum games. Section 6 concludes. All proofs can be found in the appendix.

2 Fictitious Play

2.1 Notation

Let (A, B) be a two-player bimatrix game where player 1 has pure strategies numbered from 1 to n, and player 2 has pure strategies $1, \ldots, m$. A is an $n\times m$ payoff matrix for player 1 and B an $m\times n$ payoff matrix for player 2. Thus, if player 1 chooses i and player 2 chooses j, the payoff to player 1 is

[1] Algebraic properties of projective maps have been used before in proofs of existence or uniqueness of limit cycles under continuous-time fictitious play, e.g. in Gaunersdorfer and Hofbauer (1995) or in Benaim et al. (2009). The only instance I'm aware of where a purely geometric (non-algebraic) argument is used in this context is the first of the two proofs of Lemma 2 in Gaunersdorfer and Hofbauer (1995), which they attribute to their colleague Gerhard Kowol from the University of Vienna and which inspired the present work.

a_{ij} and the payoff to player 2 is b_{ji}. The set of mixed strategies of player 1 is then the $n-1$ dimensional probability simplex S_n, and analogously S_m is the set of mixed strategies of player 2. The expected payoff for player 1 using strategy i if player 2 uses the mixed strategy $\mathbf{q} \in S_m$ is $(A\mathbf{q})_i$. Analogously $(B\mathbf{p})_j$ is the expected payoff for player 2 using strategy j against the mixed strategy $\mathbf{p}$. If both players use mixed strategies $\mathbf{p}$ and $\mathbf{q}$, respectively, the expected payoffs are $\mathbf{p}'A\mathbf{q}$ to player 1 and $\mathbf{q}'B\mathbf{p}$ to player 2, where the prime denotes the transpose of a vector. For $\mathbf{p} \in S_n$ we denote by $BR(\mathbf{p})$ the set of best responses to $\mathbf{p}$. A pair of mixed strategies $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium if and only if $\mathbf{q}^* \in BR(\mathbf{p}^*)$ and $\mathbf{p}^* \in BR(\mathbf{q}^*)$.

For $t = 1, 2, 3, \ldots$, the sequence $(\mathbf{p}(t), \mathbf{q}(t))$ is a *discrete fictitious play process*, if

$$(\mathbf{p}(1), \mathbf{q}(1)) \in S_n \times S_m$$

and for all t,

$$\mathbf{p}(t+1) = \frac{t\mathbf{p}(t) + \mathbf{b}^1(t)}{t+1}, \text{ and } \mathbf{q}(t+1) = \frac{t\mathbf{q}(t) + \mathbf{b}^2(t)}{t+1},$$

where $\mathbf{b}^1(t) \in BR(\mathbf{q}(t))$ and $\mathbf{b}^2(t) \in BR(\mathbf{p}(t))$.[2]

2.2 Continuous Fictitious Play and the Best Response Dynamics

If we go from discrete time steps to continuous time $t > 0$, we obtain the *continuous fictitious play process* (CFP)

$$\dot{\mathbf{p}}(t) = \frac{\mathbf{b}^1(t) - \mathbf{p}(t)}{t}, \qquad \dot{\mathbf{q}}(t) = \frac{\mathbf{b}^2(t) - \mathbf{q}(t)}{t}.$$

Up to a rescaling of time - which does not change the shape of the orbits - CFP is equivalent to the *best response dynamics*

$$\dot{\mathbf{p}}(t) = \mathbf{b}^1(t) - \mathbf{p}(t), \qquad \dot{\mathbf{q}}(t) = \mathbf{b}^2(t) - \mathbf{q}(t).$$

The best response dynamics was introduced by Gilboa and Matsui (1991) and Matsui (1992). The usual interpretation is based on the population model of game dynamics. In this interpretation, each player sticks to some pure strategy, but every now and then a small fraction of players may revise their strategy and switch to a pure best response to the average mixed strategy in the opponent population.

Obviously, if some pure strategy i is the *unique* best response to $\mathbf{q}(0)$, the CFP path $\mathbf{p}(t)$ is a straight line, heading for i, as long as this strategy remains the unique best response[3]. The sets of states $\mathbf{q}$ with $BR(\mathbf{q}) = \{i\}$

[2] Berger (2007a) discusses Brown's (1951) original formulation of the process, which deviates from the standard version presented here.

[3] More generally, Hofbauer (1995) shows that for any initial condition a piecewise linear best response or CFP path can be constructed.

for different pure strategies i are disjoint, open and convex subsets of S_m. If at time t_1 the best response changes to i', then the path $\mathbf{p}(t)$ suddenly heads for strategy i'. At the turning point $\mathbf{p}(t_1)$ the respective payoffs are equal. The same holds analogously for the path $\mathbf{q}(t)$. If $(\mathbf{p}(t), \mathbf{q}(t))$ converges to a point $(\mathbf{p}^*, \mathbf{q}^*)$ for $t \to \infty$, then this point is a Nash equilibrium of the game.

The best response dynamics can be applied to the single population model of evolutionary game theory, where pairs of players are drawn from the same population to play a symmetric game. The analogous model for CFP demands that both players play a symmetric game *and* use the same initial move (and hence the same moves throughout the play). To avoid confusion, we refer to the latter interpretation if we speak of CFP paths in symmetric games.

2.3 Cyclic Play

Consider a CFP path $\mathbf{x}(t)$ which has the property that there is a sequence of times $(t_0, t_1, t_2, \ldots)$, such that for $k \geq 1$, $BR(\mathbf{x}(t))$ is a singleton for all $t \in (t_{k-1}, t_k)$. The times t_k are the times where (at least) one of the players switches to another best response. We call the (pairs of) mixed strategies $\mathbf{x}(t_k)$ the *switching-points* of the path. Let the pure strategy (pair) $\mathbf{b}(t_k)$ be the goal of the CFP path at times $t \in (t_{k-1}, t_k)$, i.e. for these t let $\mathbf{b}(t_k) = BR(\mathbf{x}(t))$. Following Krishna and Sjöström (1998), we define a cyclic path[4] as a path along which the same sequence of best responses is repeated over and over. Formally, we call a CFP path $\mathbf{x}(t)$ *cyclic*, if there is a sequence of $R \geq 2$ (pairs of) pure strategies $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_R$, such that for $k \geq 1$, $k \bmod R = r$ implies $\mathbf{b}(t_k) = \mathbf{b}_r$.

If player 1, say, switches from i_1 to i_2, then at the switching time t_1 he is indifferent between these two pure strategies - both do equally well (and at least as good as his other pure strategies) against $\mathbf{q}(t_1)$. Geometrically, the set of points $\mathbf{q} \in S_m$ where player 1 is indifferent between two given pure strategies i_1 and i_2, is a linear subspace of S_m. The next switch again occurs when one of the players is indifferent between two of his pure strategies. Between switches, the CFP path is a straight line heading for some pair of pure best responses. The analogous considerations hold for symmetric games. This implies that all the maps sending some switching-point to the next switching-point are *perspectivities* (for an exact definition see below). The composition of the consecutive perspectivities along a cycle define a Poincaré map for the CFP path. Poincaré maps often turn out to be useful tools in analyzing the behavior of orbits of continuous dynamical systems. They reduce the dimension of the state space and simultaneously yield a discrete dynamical system which is often easier to study[5]. In order to analyze

[4] Rosenmüller (1971) calls it a *quasi-periodic* path.

[5] Due to the discontinuity of the vector field, linearization around an equilibrium is not possible for CFP, and one must either apply topological tools, as Ljapunov functions, or, as in this paper, study the properties of a suitable Poincaré map.

this Poincaré map we have to take a closer look at the perspectivities it is built of. This is the purpose of the next section.

3 Projective Geometry

Consider the Euclidean plane $\mathbb{R}^2$. Two distinct and non-parallel lines intersect in exactly one point. If the lines are parallel, however, their intersection is empty. In projective geometry, this "exception" is defined away by extending the Euclidean plane in a way which can intuitively be described as follows: Given a line l in $\mathbb{R}^2$, the set consisting of l and all lines parallel to l is called a *pencil of parallel lines.* To each pencil of parallel lines we add an abstract entity P_∞ called the *point at infinity* of the pencil. The lines of the pencil are said to intersect at P_∞. ("Parallels meet at infinity.") The set of all points at infinity form the *line at infinity.* The union of the Euclidean plane and the line at infinity is called the *real projective plane* and denoted by Π. In Π two distinct points (one or both of which might be at infinity) define exactly one line, and two distinct lines (one of which might be at infinity) meet in exactly one point (which is at infinity in the case of parallels). This is a standard method in projective geometry as explained e.g. in Casse (2006), to whom the interested reader is referred to for more details.[6]

3.1 Pappus' Theorem

We introduce the following notation:

For two distinct points A and B in Π we denote the line spanned by this points by AB, and the intersection of two lines (or, more generally, two nonempty sets) a and b in Π is denoted by $a \cap b$. Three or more lines are said to be *concurrent*, if they have one point in common, and three or more points are said to be *collinear*, if they all lie on one line. We will now state an important and well-known theorem which we need later on.

Theorem 1 (Pappus) *Let l and l' be distinct lines in the real projective plane. Let A, B, C be three distinct points on l and A', B', C' three distinct points on l', all different from $l \cap l'$. Define $P := AB' \cap A'B$, $Q := AC' \cap A'C$, $R := BC' \cap B'C$. Then P, Q, R are collinear.*

Pappus' theorem is illustrated in Figure 1.

[6] This intuitive method can easily be formalized by introducing homogeneous coordinates, but we refrain from doing so here since we don't need those for our definitions and results.

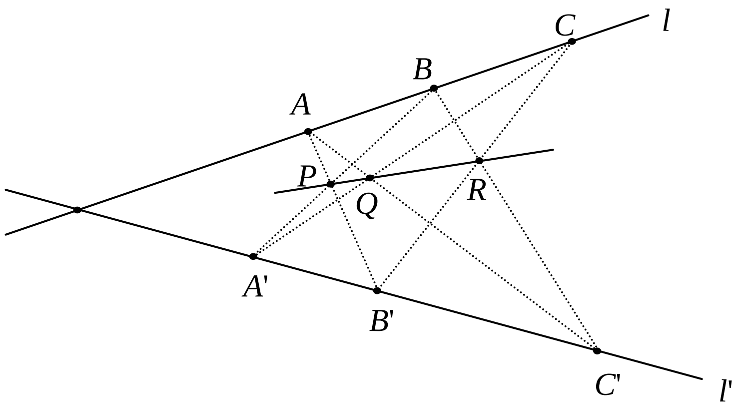

Fig. 1 Pappus' theorem.

3.2 Perspectivities in Real Projective n-Space

We constructed the real projective plane by adding to $\mathbb{R}^2$ the points at infinity, forming the line at infinity. Analogously we can construct the *real projective 3-space* by adding to $\mathbb{R}^3$ the points and lines at infinity, which build up the *plane at infinity*. For higher dimensions, the *real projective n-space* is constructed in the same way. We write $P^n(\mathbb{R})$ for the real projective n-space, hence $\Pi = P^2(\mathbb{R})$.

A subset T of $P^n(\mathbb{R})$ is called a *subspace* if for any two points $A, B \in T$ also the line AB belongs to T. The subspaces of $P^n(\mathbb{R})$ are simply the Euclidean subspaces of $\mathbb{R}^n$, extended by the respective elements at infinity. A k-dimensional subspace is called a *k-plane*. In the special case $k = n-1$ we speak of a *hyperplane* in $P^n(\mathbb{R})$. It can be shown that a k-plane U and an m-plane V in general position in $P^n(\mathbb{R})$ have nonempty intersection $U \cap V$ if $k + m \geq n$, and the intersection is a $(k+m-n)$-plane. Specifically, the formula

$$\dim(U \cap V) = \dim U + \dim V - \dim(U + V),$$

known from the theory of vector spaces, holds, where $U + V := \{X \in YZ : Y \in U, Z \in V\}$ denotes the span of U and V.

We call a set of points in $P^n(\mathbb{R})$ *co-k-planar*, if the points span a k-plane. For $k = n-1$ we also write *co-hyperplanar*. Co-1-planar thus means collinear.

Definition 1 *Let H_1 and H_2 be two distinct hyperplanes in $P^n(\mathbb{R})$. Let π be a map from H_1 to H_2. π is a* perspectivity*, if there exists a point Z, not*

on either hyperplane, such that for any $X \in H_1$, $\pi(X) = XZ \cap H_2$.
Z is called the center *of the perspectivity, and $H_1 \cap H_2$ the* axis *of the perspectivity.*

A perspectivity is bijective and the inverse of this map is also a perspectivity. Note that the axis of π is an $(n-2)$-plane which is pointwise fixed under π. The following lemma states a well known result from projective geometry about the composition of perspectivities.

Lemma 1 *The composition $\pi := \pi_2 \circ \pi_1$ of two perspectivities $\pi_1 : H_1 \to H_2$ with center Z_1 and $\pi_2 : H_2 \to H_3$ with center Z_2 between pairwise distinct hyperplanes, where π_1 and π_2 have the same axis $H_1 \cap H_2 = H_2 \cap H_3$, is again a perspectivity. The center of this perspectivity lies on the line Z_1Z_2.*

By induction it follows that the composition of finitely many perspectivities with the same axis, mapping some hyperplane H_1 to some hyperplane H_m, is a perspectivity, if $H_1 \neq H_m$. If $H_m = H_1$, i.e., if the mapping returns to H_1, the 'return map' on H_1 can thus always be written as the composition of two perspectivities, viz. $H_m \circ (H_{m-1} \circ \cdots \circ H_1)$. The next section gives a precise definition of these return maps.

3.3 Perspective Collineations

Definition 2 *A* perspective collineation *σ on a hyperplane H_1 is a composition $\pi_2 \circ \pi_1$ of two perspectivities $\pi_1 : H_1 \to H_2$ and $\pi_2 : H_2 \to H_1$ with centers Z_1 and Z_2. The* axis *of σ is the axis $L := H_1 \cap H_2$ of π_1 (and π_2). For $Z_1 \neq Z_2$, the* center *of σ is the point $Z := Z_1Z_2 \cap H_1$.*
If $Z \in L$, then σ is called an elation*, otherwise it is called a* homology.

In the degenerate case where the centers Z_1 and Z_2 coincide, the perspective collineation is the identity on H_1. If the perspective collineation σ is not degenerate, then the center and all the points on the axis are fixed points of σ, and there are no other fixed points (see Figure 2 for an illustration in the real projective plane). Hence, if σ is an elation, the points on the axis are the only fixed points. This is important for the study of the dynamics resulting from iteration of a perspective collineation on a hyperplane. In the case of an elation, there are no fixed points apart from those on the axis. This rules out the existence of periodic orbits of CFP, since the intersection of a periodic orbit with a hyperplane would be a fixed point under the respective return map. In the next theorem we provide sufficient conditions for σ to be an elation.

Theorem 2 *In $P^n(\mathbb{R})$ for $n \geq 2$ let $\sigma : H_1 \to H_1$ be the composition $\pi_k \circ \cdots \circ \pi_1$ of finitely many perspectivities $\pi_1 : H_1 \to H_2, \ldots, \pi_k : H_k \to H_1$ ($k \geq 3$) between hyperplanes H_i with common axis $L \subset H_1$ and centers $Z_1, \ldots, Z_k$ in general position*[7]*. If the intersection points $F_1 := Z_kZ_1 \cap$*

[7] Points $Z_1, \ldots, Z_k$ in $P^n(\mathbb{R})$ are in general position, if no $n+1$ of them are co-hyperplanar.

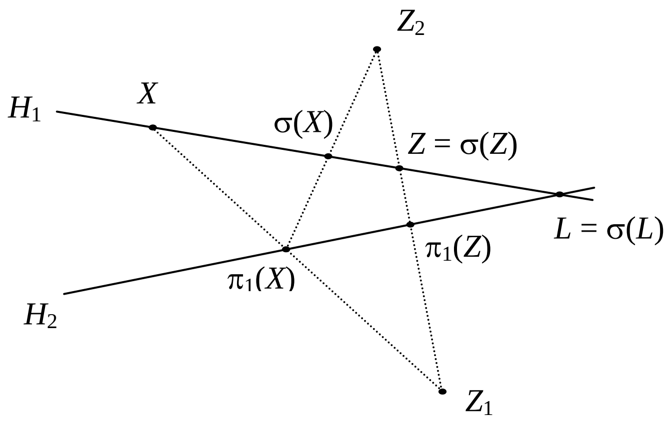

Fig. 2 The axis L and the center Z of a perspective collineation composed of two perspectivities in $P^2(\mathbb{R})$.

H_1, $F_2 := Z_1Z_2 \cap H_2, \ldots, F_k := Z_{k-1}Z_k \cap H_k$ are co-hyperplanar, then σ is an elation.

Proof See appendix.

To study the stability of fixed points of a perspective collineation, it is most convenient to return from a projective space to a Euclidean space by removing the elements at infinity. A perspective collineation induces a map on this Euclidean space. This map is well defined except on points which map to points at infinity under the collineation.[8] It can be shown that the center of a homology (if it is not at infinity) is always a hyperbolic fixed point of the induced map. The center of an elation is of course non-hyperbolic, since it lies on the pointwise fixed axis. As the next lemma shows, the behavior of orbits near the center of an elation follows a simple pattern.

Lemma 2 *Consider a perspective collineation σ on a hyperplane H_1 with center Z. Let $\bar{H}_1$ be the Euclidean hyperplane constructed by removing the elements at infinity from H_1, and let $\bar{\sigma}$ be the return map on $\bar{H}_1$ induced by σ. If σ is an elation, Z is semistable under iterations of $\bar{\sigma}$, i.e., on one side of the axis, all orbits converge to Z, and on the other side, all orbits diverge from Z.*

Proof See appendix.

[8] E.g. in Figure 2, if X is moved to the right along H_1 until the line XZ_1 is parallel to H_2, then X has no image under the induced map π_1 *in Euclidean space* and therefore also $\sigma(X)$ is no longer defined at this point.

4 The Main Result

Now we have the necessary instruments to bridge the gap between projective geometry and CFP. If a CFP path is cyclic, then it defines a Poincaré or return map on an indifference hyperplane. Geometrically, the return map is a composition of perspectivities. In Theorem 2 we have found conditions under which the return map is an elation, i.e., a perspective collineation whose axis contains its center. Translating these conditions into properties of a game reveals that the center is a Nash equilibrium in this case, and applying Lemma 2 yields the semistability of this equilibrium under the return map. Since the return map is suitably defined only on one side of the equilibrium, either all orbits converge to the equilibrium, or all orbits diverge from the equilibrium. Divergence, however, is ruled out by the observation that the return map points inwards at the boundary of the mixed strategy space. Hence the equilibrium attracts all cyclic CFP paths outside the equilibrium set. This result is made precise by Theorem 3 below. We start by defining a property of a game that allows us to make use of Theorem 2 and leads us to the main result.

Definition 3 *A symmetric game has the* parallel property*, if every hyperplane of indifference*[9] *between two pure best responses i and j is parallel in the space of mixed strategies to the line connecting i and j.*
A bimatrix game has the parallel property*, if every hyperplane of indifference of player 1 (player 2) between two pure best responses i and j is parallel in the space of mixed strategy profiles to the line connecting $(i, \mathbf{b})$ and $(j, \mathbf{b})$ ($(\mathbf{b}, i)$ and $(\mathbf{b}, j)$), where $\mathbf{b}$ is an arbitrary pure best response of player 2 (player 1).*

Remark: Any symmetric zero-sum game A has the parallel property. This is because the hyperplane of indifference is the set $\{\mathbf{x} \in S_n : (\mathbf{e}_i - \mathbf{e}_j)' A\mathbf{x} = 0\}$, and the vector $\mathbf{e}_i - \mathbf{e}_j$ is parallel to this hyperplane, since the zero-sum property of A implies $(\mathbf{e}_i - \mathbf{e}_j)' A(\mathbf{e}_i - \mathbf{e}_j) = 0$. Note, that the other direction is not true. There are symmetric games with the parallel property, which are not zero-sum. An example will be described in section 4.3.

For bimatrix games (A, B) we note that *every bimatrix game has the parallel property.* The reason for this is that the hyperplane of indifference of player 1 between i and j is the set $\{(\mathbf{x}, \mathbf{y}) \in S_n \times S_m : (\mathbf{e}_i - \mathbf{e}_j)' A\mathbf{y} = 0\}$, which is trivially parallel to the vector $(\mathbf{e}_i, \mathbf{b}) - (\mathbf{e}_j, \mathbf{b}) = (\mathbf{e}_i - \mathbf{e}_j, \mathbf{0})$, and the same holds for player 2, independently of $i, j, \mathbf{b}$, and the payoff matrices.

Theorem 3 *Let G be a finite two-person game with the parallel property and a set of completely mixed Nash equilibria with codimension 2. Then there is a single Nash equilibrium that attracts all cyclic CFP paths.*

[9] In finite two-person games, because of the linearity of payoffs, the set of points where a player is indifferent between two given pure strategies is generically the intersection of this player's mixed-strategy simplex with a hyperplane. This intersection is called the *hyperplane of indifference* corresponding to the two strategies.

Proof Fix some hyperplane of indifference H_1, where switching occurs for CFP paths. Any other hyperplane of indifference distinct from H_1 cuts H_1 into 2 halves. Switching occurs only on one half of H_1, the half where the pure strategies that the player is indifferent between, are best responses. Any cyclic path crosses this half-hyperplane infinitely often. Define σ as the return map on this half of H_1. The maps sending one switching point to the next are perspectivities with a common axis, since all the hyperplanes of indifference intersect in a set L of codimension 2, which is therefore identical to the set of completely mixed Nash equilibria of the game. The centers of these perspectivities are (pairs of) pure strategies, and are therefore in general position. The parallel property of G implies that the lines through two consecutive (pairs of) best responses intersect the correponding hyperplanes of indifference at the hyperplane at infinity, hence all these intersection points are co-hyperplanar. Applying Theorem 2 tells us that σ is an elation, and by Lemma 2, we conclude that on the half-hyperplane where σ is defined, either all orbits of the return map converge to its center, or all orbits diverge from its center. The second possibility is ruled out because the return map points inwards at the boundary of the space of (pairs of) mixed strategies. It follows that any cyclic CFP path outside the equilibrium set converges to the center of σ, which is a particular Nash equilibrium of G. □

4.1 Example 1: Matching Pennies

The well known game of Matching Pennies is the standard representative of the class of 2×2 bimatrix games with a unique and completely mixed equilibrium. Any such game is strategically equivalent to a zero-sum game (Hofbauer and Sigmund, 1998). Its cyclic best response structure implies that every nonconstant CFP path is cyclic (see e.g. Rosenmüller, 1971, or Metrick and Polak, 1994), as sketched in Figure 3(a). Since the game of Matching Pennies is zero-sum, convergence to equilibrium follows from Robinson's (1951) result. However, it can also be derived from Theorem 3, since the game has the parallel property and the equilibrium has codimension 2 (it is a point in a plane).

4.2 Example 2: Rock-Scissors-Paper

The Rock-Scissors-Paper game, short RSP game, is a symmetric two-person game where each player has pure strategies *rock*, *scissors*, and *paper*, and the best response structure is cyclic: rock beats scissors, scissors beat paper, and paper beats rock. With payoffs 1, 0, and -1 for win, draw, and loss, respectively, this game is zero-sum[10], and all nonconstant CFP paths are

[10] CFP for generalized RSP games is treated in Gaunersdorfer and Hofbauer (1995).

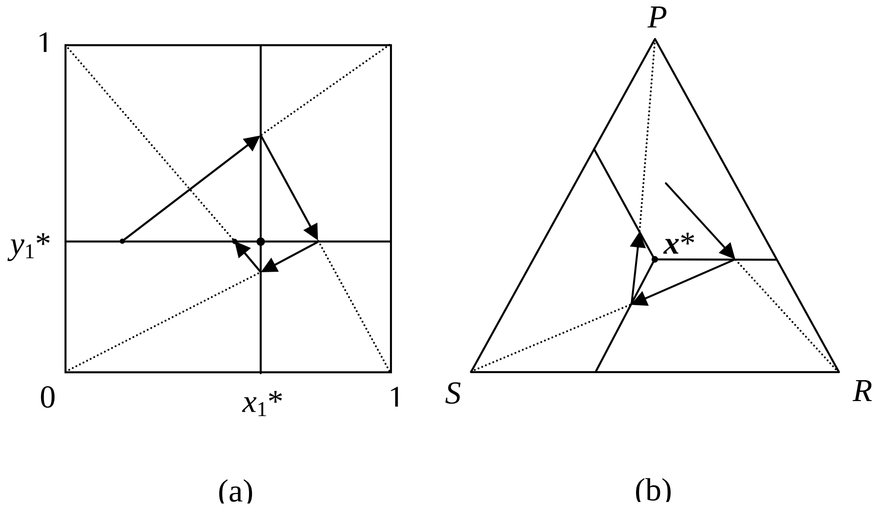

Fig. 3 CFP paths for a Matching Pennies game (a) and the Rock-Scissors-Paper game (b).

cyclic (see Figure 3(b)). Again convergence to the completely mixed equilibrium $(1/3, 1/3, 1/3)'$ follows, because the game is zero-sum, but is also implied by Theorem 3, since the RSP game has the parallel property and a completely mixed equilibrium of codimension 2.

4.3 Example 3: 2×2 *Role Games*

Given some bimatrix game called the *base game*, the corresponding *role game* is defined as the symmetric game where each player is assigned the role of player 1 and player 2 with probabilities 1/2 each, and then plays the base game against the other player. This definition implies a special structure for the payoffs in the role game. If the base game is of the Matching Pennies type as in our first example, then it can be shown (see Berger, 2001), that the role game is a 4×4 game with a line segment of completely mixed equilibria, and all CFP paths outside this equilibrium line are cyclic. Moreover, this role game has the parallel property, but in general it is not zero-sum. In Berger (2001), it is proved that there exists a unique equilibrium that attracts all CFP paths outside the equilibrium line. There, the attracting equilibrium is explicitly calculated. However, the mere existence of such an equilibrium is a direct consequence of Theorem 3.

5 Symmetric 4×4 Zero-Sum Games

The application of the main theorem to these examples provides new proofs for known results. Indeed, convergence of CFP for games that are strategically equivalent to zero-sum games (as in the first two examples) is most conveniently proved by invoking an appropriate Ljapunov function, see Hofbauer (1995). If a zero-sum game has an isolated equilibrium, then this equilibrium is unique. However, for some classes of zero-sum games there is no isolated equilibrium, but only a continuum of Nash equilibria. For these cases, Ljapunov's argument shows convergence of CFP paths to the set of equilibria, but it does not tell us anything about how the paths approach this set. It could be the case that any equilibrium attracts some path, or that only a strict subset of the equilibrium set is attractive. The first of these alternatives sounds more plausible, since from the viewpoint of classical, static game theory, there is no way to discriminate among the different equilibria – they are *equivalent*[11] in the sense of van Damme (1983). However, we will show below, that under CFP, or in a population of best response players, a *particular* single equilibrium may emerge in the long run.

For a symmetric game A which is strategically equivalent to a zero-sum game, we have $A = -A'$ without loss of generality, and $\mathbf{x}'A\mathbf{x} = 0$ for all $\mathbf{x} \in S_n$. If there is a completely mixed equilibrium $\mathbf{x}^*$, then $A\mathbf{x}^* = \mathbf{0}$. Therefore the kernel of A is nontrivial, at least one eigenvalue of A is 0. Since A is skew-symmetric, its eigenvalues are purely imaginary, appearing in complex conjugate pairs. The set of completely mixed Nash equilibria is $\text{int}S_n \cap \text{Ker}A$, the intersection of the interior of S_n with the kernel of A in $\mathbb{R}^n$. The dimension of this intersection is $m-1$, where m is the dimension of $\text{Ker}A$, which equals the multiplicity of the eigenvalue 0. Hence the completely mixed equilibrium can only be unique, if the eigenvalue 0 has multiplicity one, what in turn implies that the number n of pure strategies in the game is odd. If n is even, the multiplicity of the eigenvalue 0 is also even, and the set of completely mixed equilibria is either empty or a continuum. For $n = 2$ an interior equilibrium exists only in the degenerate case, where the whole simplex consists of equilibria. The case $n = 3$ was subject of our second example, the RSP game. If a completely mixed equilibrium exists for $n = 4$, then it is contained in a continuum L of equilibria in S_4. The dimension of L is either 3 (the degenerate case) or 1.

If the dimension of L is 1, its codimension is 2. Hence Theorem 3 applies and we can formulate

Proposition 1 *For any symmetric* 4×4 *zero-sum game there exists one particular equilibrium which attracts all CFP paths outside the equilibrium set.*

[11] Two equilibria $\mathbf{p}$ and $\mathbf{q}$ of a symmetric game A are equivalent, if $\mathbf{x}'A\mathbf{p} = \mathbf{x}'A\mathbf{q}$ and $\mathbf{p}'A\mathbf{x} = \mathbf{q}'A\mathbf{x}$ for any $\mathbf{x} \in S_n$.

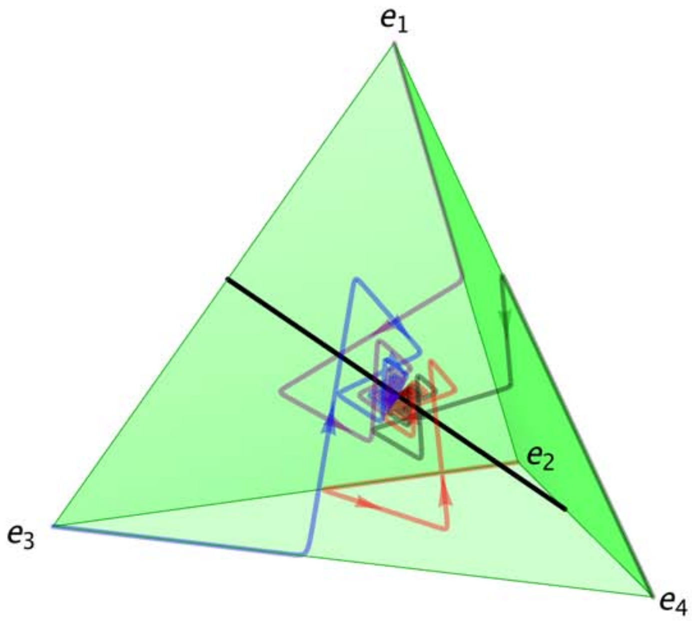

Fig. 4 Four CFP paths for a 4×4 zero-sum game with a line segment of Nash equilibria (black line). All CFP paths converge to the barycenter of the simplex.

Note that this can be read as an equilibrium selection result. Indeed, CFP (with identical initial move) or the best response dynamics select a particular equilibrium among a continuum of equivalent equilibria as the long run outcome.

As an illustration, consider the symmetric zero-sum game

$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix},$$

taken from Sandholm (2010, p. 321). This game has a line segment of Nash equilibria connecting the states $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$. Figure 4 shows the four CFP paths starting at the vertices, all converging to the single equilibrium $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

6 Summary

We introduced the discrete fictitious play process, continuous-time fictitious play, and the equivalent best response dynamics. We defined cyclic play and argued that since CFP paths are typically the union of line segments pointing at pure strategies in the state space, the theory of projective geometry might provide useful tools for the analysis of convergence of cyclic paths. After establishing the necessary results from projective geometry, we defined the parallel property for games and showed that this property, together with a Nash equilibrium set of codimension 2 implies the global convergence of cyclic CFP paths to a particular Nash equilibrium. This main result was illustrated by three examples, and as a first corollary we presented a new result which can be interpreted as an equilibrium selection result for symmetric 4×4 zero-sum games. Admittedly, this is a rather specific result, but the present note mainly intends to introduce the techniques of projective geometry, and we hope that more general convergence or nonconvergence results can be produced by these methods in the future.

7 Appendix

7.1 Proof of Theorem 2

For the proof of Theorem 2 we use induction with respect to n, the dimension of the projective space. The induction basis, the case $n = 2$, is given by the following lemma, which is proved separately.

Lemma 3 *In the real projective plane, let $\sigma : l_1 \to l_1$ be the composition $\pi_k \circ \cdots \circ \pi_1$ (where $k \geq 3$) of finitely many perspectivities between concurrent lines, i.e., perspectivities $\pi_1 : l_1 \to l_2, \ldots, \pi_k : l_k \to l_1$ with common axis $L \in l_1$ and centers $Z_1, \ldots, Z_k \in \Pi$, no three of which are collinear. If the intersection points $F_1 := Z_k Z_1 \cap l_1$, $F_2 := Z_1 Z_2 \cap l_2, \ldots, F_k := Z_{k-1} Z_k \cap l_k$ are collinear, then σ is an elation.*

Proof We prove this lemma by induction with respect to k, first showing that it holds for $k = 3$. This case is illustrated in Figure 5. By Lemma 1, $\pi_2 \circ \pi_1$ is again a perspectivity with its center Z_{12} lying on $Z_1 Z_2$. We can construct this center in the following way: Let F_1' be the image of F_1 under π_1, and let F_1'' be the image of F_1' under π_2, then F_1'' is also the image of F_1 under $\pi_2 \circ \pi_1$. Formally

$$F_1' := F_1 Z_1 \cap l_2 , \quad F_1'' := F_1' Z_2 \cap l_3 , \quad Z_{12} = F_1 F_1'' \cap Z_1 Z_2 .$$

Note that due to the collinearity of Z_1, Z_2, and F_2, we can also see that $Z_{12} = F_1 F_1'' \cap F_2 Z_2$. The points F_1, F_2, F_3 are collinear by assumption, and Z_2, F_1'', F_1' are collinear by definition of F_1''. Applying Pappus' theorem to these six points yields the collinearity of the points

$$Z_{12} = F_1 F_1'' \cap F_2 Z_2 , \quad Z_3 = F_1 F_1' \cap F_3 Z_2 , \quad L = F_2 F_1' \cap F_3 F_1'' .$$

Hence for the center Z of σ (given by $Z := Z_{12} Z_3 \cap l_1$) considered above we have $Z = L$. This proves the case $k = 3$.

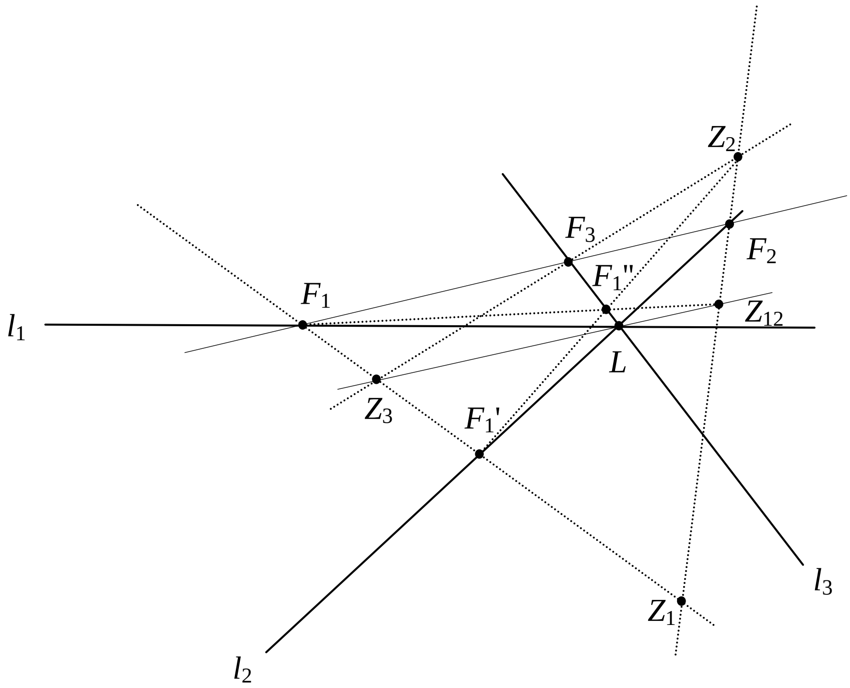

Fig. 5 The case $k = 3$ for Lemma 3.

For the induction step we show that the case $k = K$ can be proved if we assume the lemma holds for $3 \leq k \leq K - 1$. To this end, we define the point $Z_0 := Z_K Z_1 \cap Z_2 Z_3$. Remember we have perspectivities

$$\pi_i : l_i \to l_{i+1} \quad \text{with center } Z_i \quad (i = 1, \ldots, K-1), \text{ and}$$
$$\pi_K : l_K \to l_1 \quad \text{with center } Z_K .$$

Now we introduce the additional perspectivity $\pi_0 : l_1 \to l_3$ with center Z_0. Let us have a look at the compositions

$$p := \pi_K \circ \cdots \circ \pi_3 \circ \pi_0 : l_1 \to l_1 \quad \text{and} \quad q := \pi_0^{-1} \circ \pi_2 \circ \pi_1 : l_1 \to l_1 ,$$

where $\sigma = p \circ q = \pi_K \circ \cdots \circ \pi_1$ is the perspective collineation we are interested in. p is an instance of $k = K - 1$ and is an elation by assumption. So we know that L, the center Z_0 of π_0, and C, which shall denote the center of $\pi_K \circ \cdots \circ \pi_3$, are collinear. The same is true for L, the center Z_{12} of $\pi_2 \circ \pi_1$, and Z_0 (which is also the center of π_0^{-1}), since q is an instance of $k = 3$. It follows that L, Z_{12}, C are collinear. Now σ is the composition of two perspectivities $\sigma = (\pi_K \circ \cdots \circ \pi_3) \circ (\pi_2 \circ \pi_1) =: \bar{p} \circ \bar{q}$, with

$$\bar{p} : l_3 \to l_1 \text{ with center } C, \text{ and } \bar{q} : l_1 \to l_3 \text{ with center } Z_{12},$$

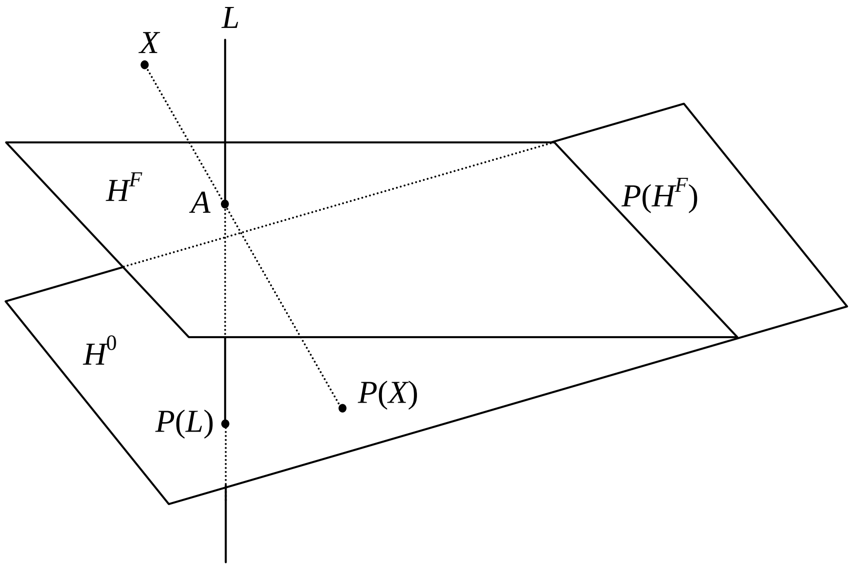

Fig. 6 Configuration in the proof of Theorem 2 for $n = 3$ and $k = 4$.

where l_1 and l_3 meet in L, and L, Z_{12}, C are collinear. For the center $Z := CZ_{12} \cap l_1$ of the perspective collineation σ this collinearity implies $Z = L$. Hence, σ is an elation. $\square$

Now that we have established the induction basis for Theorem 2, assume that the theorem holds for dimension $n-1$ and consider the case of dimension n. For this induction step we introduce a central projection P in $P^n(\mathbb{R})$ for $n \geq 3$: Let H^F be the hyperplane spanned by the points F_i $(i = 1, \ldots, k)$. Let A be an arbitrary point in $H^F \cap L$ (this intersection is nonempty, since its dimension is $n-3 \geq 0$), and choose another hyperplane H^0, which does neither contain the center Z of σ, nor A. Now define P by

$$P: \; P^n(\mathbb{R}) - \{A\} \to H^0, \quad P(X) := XA \cap H^0 \, .$$

P projects the centers Z_i and the points F_i in $P^n(\mathbb{R})$ to points $P(Z_i), P(F_i)$ in H^0. Since A is contained in the axis L, the image $P(L)$ of the axis is the intersection of L and H^0, which is an $(n-3)$-plane, while the hyperplanes H_i are mapped to $(n-2)$-planes $P(H_i)$, intersecting in $P(L)$. Now by construction of P the hyperplane H^F, containing the points F_i and A, is mapped to an $(n-2)$-plane $P(H^F)$, containing the images $P(F_i)$. In Figure 6 the situation is sketched for $n = 3$ and $k = 4$.

This projection thus simply reduces the dimension by one: H^0 can be identified with $P^{n-1}(\mathbb{R})$, and the corresponding hyperplanes of this projective space are the images $P(H_i)$ and $P(H^F)$. P induces perspectivities

$\pi_i^P : P(H_i) \to P(H_{i+1}), \quad \pi_i^P(X) = P(\pi_i(X))$ (these are well defined, since $X \in P(H_i)$ implies $X \in H_i$) with centers $P(Z_i)$ and common axis $P(L)$ between hyperplanes of H^0, and the composition of these perspectivities is the induced perspective collineation σ^P on $P(H_1)$. The axis of σ^P is $P(L)$, and its center is $P(Z)$. Certainly σ^P is an elation if and only if σ is an elation. In H^0 the points $P(F_i)$ are co-hyperplanar, and by induction assumption σ^P indeed is an elation. Hence also σ is an elation. □

7.2 Proof of Lemma 2

For any point $X \in H_1$, the three points X, $\sigma(X)$, and Z are collinear. Hence iterating σ leaves the line XZ invariant. Let σ' be the restriction of σ to this line. The center Z is the only fixed point of σ'. We can identify the line XZ with $\mathbb{R} \cup \{\pm\infty\}$, assuming w.l.o.g. $Z = \pm\infty$. Then the restriction of σ' to the real line is a continuous function admitting no fixed point. Hence either $\sigma'(x) < x$ or $\sigma'(x) > x$ for all $x \in \mathbb{R}$. In both cases all real points move in the same direction under iterations of σ'. That means, they approach Z from one side and diverge from Z from the other side. By continuity of the return map $\bar{\sigma}$, all points on one side of the axis converge to Z, and all points on the other side diverge from Z. □

References

1. M. Benaim, J. Hofbauer and E. Hopkins, Learning in games with unstable equilibria, *J. Econ. Theory* **144** (2009), 1694-1709.
2. U. Berger, Best response dynamics for role games, *Int. J. Game Theory* **30** (2001), 527-538.
3. U. Berger, Fictitious play in 2×n games, *J. Econ. Theory* **120** (2005), 139-154.
4. U. Berger, Brown's original fictitious play, *J. Econ Theory* **135** (2007a), 572-578.
5. U. Berger, Two more classes of games with the continuous-time fictitious play property, *Games Econ. Behav.* **60** (2007b), 247-261.
6. U. Berger, Learning in games with strategic complementarities revisited, *J. Econ. Theory* **143** (2008), 292-301.
7. G. W. Brown, Iterative solution of games by fictitious play, *in* "Activity Analysis of Production and Allocation" (T.C. Koopmans, Ed.), pp. 347-376, New York, Wiley, 1951.
8. R. Casse, "Projective Geometry: An Introduction," Oxford Univ. Press, Oxford, UK, 2006.
9. E. van Damme, "Refinements of the Nash Equilibrium Concept," Springer, Berlin, 1983.
10. A. Gaunersdorfer and J. Hofbauer, Fictitious play, Shapley polygons, and the replicator equation, *Games Econ. Behav.* **11** (1995), 279-303.
11. I. Gilboa and A. Matsui, Social stability and equilibrium, *Econometrica* **59** (1991), 859-867.
12. M. Hahn, Shapley polygons in 4×4 games, *Games* **1** (2010), 189-220.
13. J. Hofbauer, Stability for the best response dynamics, mimeo, Vienna, 1995.

14. J. Hofbauer and W. H. Sandholm, Stable games and their dynamics, *J. Econ. Theory* **144** (2009), 1665-16693.
15. J. Hofbauer and K. Sigmund, "Evolutionary Games and Population Dynamics," Cambridge Univ. Press, Cambridge, UK, 1998.
16. V. Krishna and T. Sjöström, On the convergence of fictitious play, *Math. Operations Res.* **23** (1998), 479-511.
17. A. Matsui, Best response dynamics and socially stable strategies, *J. Econ. Theory* **57** (1992), 343-362.
18. A. Metrick and B. Polak, Fictitious play in 2×2 games: a geometric proof of convergence, *Econ. Theory* **4** (1994), 923-933.
19. K. Miyasawa, On the convergence of the learning process in a 2×2 non-zero-sum two person game, Economic Research Program, Princeton University, Research Memorandum No. 33, 1961.
20. D. Monderer and L. S. Shapley, Fictitious play property for games with identical interests, *J. Econ. Theory* **68** (1996), 258-265.
21. J. Robinson, An iterative method of solving a game, *Ann. Math.* **54** (1951), 296-301.
22. J. Rosenmüller, Über Periodizitätseigenschaften spieltheoretischer Lernprozesse, *Z. Wahrsch. verw. Gebiete* **17** (1971), 259-308.
23. W. H. Sandholm, "Population Games and Evolutionary Dynamics," MIT Press, Cambridge, MA, 2010.
24. W. H. Sandholm, E. Dokumaci, and F. Franchetti. Dynamo: Diagrams for Evolutionary Game Dynamics, version 1.0, 2011. http://www.ssc.wisc.edu/ whs/dynamo.
25. L. S. Shapley, Some topics in two-person games, *Ann. Math. Stud.* **5** (1964), 1-28.
26. S. van Strien, Hamiltonian flows with random-walk behaviour originating from zero-sum games and fictitious play, *Nonlinearity* **24** (2011), 1715-1742.