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1 Introduction

The cumulative sum (CUSUM) technique is well-established in theory and practice of process control. A comprehensive exposition of the method is given, e.g., by Wetherill and Brown (1991). A question that is seldom treated in the literature is that on the effect of serial correlation of the control variable. Johnson and Bagshaw (1974) investigate the effect of correlation on the run length distribution when the control variable follows a first order autoregressive or moving average process. They also give an approximate expression for the average run length of the CUSUM-technique for correlated control variables.

In this paper we derive an exact expression for the average run length of a discretized CUSUM-technique, i.e., a technique that uses a scoring system for the observations of the control variable. The scoring system is that suggested by Munford (1980). Our results are derived for a control variable that is assumed to follow a first order autoregressive process and with normally distributed disturbances. After deriving in Section 2 the expression for the average run length we discuss its dependence on the process parameter and give a numerical illustration. In Section 3 we discuss corrections for the CUSUM-technique in order to keep the nominal risk for an out-of-control decision and compare our results with those given by Johnson and Bagshaw (1974).

2 The Discretized CUSUM-Technique for Correlated Control Variable

The CUSUM technique we discuss in this paper is based on scores that are obtained by trichotomizing the observations of the control variable. Such a control technique has been suggested by Munford (1980) as a generalization of a similar one proposed by Page (1955). However, our approach generalizes the usual set of assumptions in that we allow the control variable to be serially correlated. This is of great practical relevance and gives insights into the robustness of the CUSUM-technique against correlation. Note that discretization has been suggested in order not only to achieve easier mathematical tractability but also to improve the distributional robustness of procedures [e.g., by McGilchrist and Woodyer (1975), Hackl and

Ledolter (1991)].

For the case of uncorrelated control variables the performance of the score-based CUSUM-technique in terms of the average run length was discussed by Munford (1980). He found that the score-based CUSUM-technique is in general slightly less sensitive than the traditional CUSUM-technique but that it is considerably more sensitive than the Shewhart technique for small deviations in the process mean.

2.1 Assumptions on the process

We assume that the control variable Y_n of a process control scheme obeys the relation

$$Y_n = \rho Y_{n-1} + \epsilon_n \,, \tag{1}$$

for $n \geq 0$ where ρ $(-1 < \rho < 1)$ is the autoregressive parameter. In the in control case both the disturbance terms ϵ_n and the starting value of the control variable Y_0 are identically and independently normally distributed

$$\epsilon_n \sim \mathrm{N}(0, \sigma_0^2)$$

$$Y_0 \sim \mathrm{N}(0, \sigma_1^2)\,;$$

the disturbance terms ϵ_n and Y_0 are independent. Under these assumptions we find for the first two moments of Y_n

$$\mathrm{E}\{Y_n\} = 0 \tag{2}$$

and

$$\mathrm{Var}\{Y_n\} = \frac{1 - \rho^{2(n-1)}}{1 - \rho^2}\sigma_0^2 + \rho^{2(n-1)}\sigma_1^2 \tag{3}$$

for $n \geq 0$. Since the disturbance terms ϵ_n are normally distributed, the process Y_n is a Markov process [see, e.g., Feller (1971), pp. 93]. For large n, the distribution of Y_n is approximately independent of n.

2.2 The discretized CUSUM technique

Following Munford (1980) we consider the discretized statistic

$$Z_n = \begin{cases} 1 & \text{if} \quad Y_n > k \\ 0 & \text{if} \quad -k \le Y_n \le k \\ -1 & \text{if} \quad Y_n < -k \end{cases} \tag{4}$$

for $n \ge 0$. In agreement with standard convention we define the discrete CUSUM statistic

$$T_n = \max_{r \le n} \left[S_r - \min_{i \le r} S_i \right] \tag{5}$$

where

$$S_n = \sum_{i=0}^{n} Z_i \, .$$

are the cumulative "scores" of the first n observations. The process is considered to be out-of-control at time n if $T_n \ge m$; the integer m is the suitably chosen control limit. It should be noted that the procedure coincides with the Shewhart technique for $m = 1$. This one-sided control procedure can be extended to a two-sided one in a straightforward way.

If $\rho = 0$ the probability function of Z_n is given by [see Munford (1980)]

$$\begin{aligned} P(Z_n = 1) &= \alpha \\ P(Z_n = 0) &= \beta \\ P(Z_n = -1) &= \gamma \end{aligned}$$

where $\alpha = 1 - \Phi(k), \gamma = \Phi(-k)$, and $\beta = 1 - \alpha - \gamma$. For the more general case where $\rho \neq 0$, we introduce the jump probabilities

$$\begin{aligned} P(Z_n = 1 | Z_{n-1} = i) &= \alpha_{i,n} \\ P(Z_n = 0 | Z_{n-1} = i) &= \beta_{i,n} \\ P(Z_n = -1 | Z_{n-1} = i) &= \gamma_{i,n} \end{aligned} \tag{6}$$

for $i = -1, 0$, and 1, where

$$\begin{aligned} \alpha_{1,n} &= P(Y_n > k | Y_{n-1} > k) \\ \alpha_{0,n} &= P(Y_n > k | -l \le Y_{n-1} \le k) \end{aligned}$$

$$\begin{aligned}\alpha_{-1,n} &= P(Y_n > k | Y_{n-1} < -k) \\ \gamma_{1,n} &= P(Y_n < -k | Y_{n-1} > k) \\ \gamma_{0,n} &= P(Y_n < -k | -k \le Y_{n-1} \le k) \\ \gamma_{-1,n} &= P(Y_n < -k | Y_{n-1} < -k)\end{aligned}$$

and $\beta_{i,n} = 1 - \alpha_{i,n} - \gamma_{i,n}$.

The dependence of these conditional probabilities on n corresponds to the more obvious dependence of the variance of Y_n on n. If

$$\sigma_1^2 = \frac{\sigma_0^2}{1-\rho^2}, \tag{7}$$

the variances of Y_n and, as a result, also the conditional probabilities (6) will be independent of n. The same is approximately true for large values of n. Consequently, the conditional probabilities (6) are (approximately) independent of n for large n and/or if (7) applies.

2.3 The effect of correlation on the average run length

The run length or stopping time is defined as

$$N_{a,i} = \inf\{n : T_n \ge m, T_0 = a, Z_0 = i\} \tag{8}$$

for suitable starting values a ($0 \le a < m$) and i ($i = -1, 0, 1$), and the control limit m of the CUSUM technique. The run length $N_{a,i}$ equals the time to absorption of the (correlated) random walk T_n, conditional on T_0 and Z_0, at state m. Note that because of the Markov property of Y_n, Z_n is itself Markovian, having the statespace $S = \{-1, 0, 1\}$ and the one-step transition matrix

$$P_n = \begin{pmatrix} \gamma_{-1,n} & \beta_{-1,n} & \alpha_{-1,n} \\ \gamma_{0,n} & \beta_{0,n} & \alpha_{0,n} \\ \gamma_{1,n} & \beta_{1,n} & \alpha_{1,n} \end{pmatrix}.$$

The correlated random walk T_n has the statespace $V = \{0, 1, 2, \ldots, m\}$, where m is an absorbing state. Although T_n is not a Markov chain, we can state a corresponding Markov chain by extending the statespace of T_n. An appropriate extension is the bivariate process $\{T_n, Z_n\}$ that has the statespace $V' = \{(0,-1),(0,0),(0,1),(1,-1),\ldots,(m-1,-1),(m-1,0),(m-1,1),(m,-1),(m,0),(m,1)\}$. The corresponding one-step transition matrix $\mathbf{Q}_n$ can be given in blocked form

$$\mathbf{Q}_n = \begin{pmatrix} \mathbf{D}_n & \mathbf{A}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{C}_n & \mathbf{B}_n & \mathbf{A}_n & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n & \mathbf{B}_n & \mathbf{A}_n & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} \end{pmatrix}.$$

All blocks are of order 3×3 and given by

$$\mathbf{D}_n = \begin{pmatrix} \gamma_{-1,n} & \beta_{-1,n} & 0 \\ \gamma_{0,n} & \beta_{0,n} & 0 \\ \gamma_{1,n} & \beta_{1,n} & 0 \end{pmatrix}, \qquad \mathbf{A}_n = \begin{pmatrix} 0 & 0 & \alpha_{-1,n} \\ 0 & 0 & \alpha_{0,n} \\ 0 & 0 & \alpha_{1,n} \end{pmatrix},$$

$$\mathbf{C}_n = \begin{pmatrix} \gamma_{-1,n} & 0 & 0 \\ \gamma_{0,n} & 0 & 0 \\ \gamma_{1,n} & 0 & 0 \end{pmatrix}, \qquad \mathbf{B}_n = \begin{pmatrix} 0 & \beta_{-1,n} & 0 \\ 0 & \beta_{0,n} & 0 \\ 0 & \beta_{1,n} & 0 \end{pmatrix}.$$

Let $\mathbf{R}_n$ denote the submatrix of $\mathbf{Q}_n$ corresponding to the transient states. From the theory of absorbing Markov chains follows that the tail probability $P(N_{a,i} > k)$ of the distribution of $N_{a,i}$ is given by the sum of the $(3a+i+2)$-th row of the matrix product $\mathbf{R}_0\mathbf{R}_1\ldots\mathbf{R}_k$. Therefore, we obtain for the average run length

$$\mathrm{E}\{N_{a,i}\} = e' \sum_{j\geq 0} \mathbf{R}_0\mathbf{R}_1\ldots\mathbf{R}_j l \tag{9}$$

where $\mathbf{R}_0$ is the identity matrix, e is a unit vector with 1 as the $(3a+i+2)$-th component and l is a $(3a+i+2)$-vector of ones.

If the conditional probabilities (6) are independent of n, i.e., if n is large, an approximate analytic expression can be given for the average run length. This expression is exact if (7) holds. Taking into account that in the in-control case $\gamma_{-1} = \alpha_1, \gamma_0 = \alpha_0$, and $\alpha_{-1} = \gamma_1$ due to the symmetry of the normal distribution, we find for the average run length $E\{N_{0,i}\}$ the explicit expression

$$\mathrm{E}\{N_{0,1}\} = \frac{m(1+2\alpha_0-\alpha_1-\gamma_1)[1+3\alpha_1-3\gamma_1+m(1-\alpha_1+\gamma_1)]}{2\alpha_0(1+\alpha_1-\gamma_1)} \tag{10}$$

$$\mathrm{E}\{N_{0,-1}\} = \frac{(1+2\alpha_0-\alpha_1-\gamma_1)}{2\alpha_0(1+\alpha_1-\gamma_1)(1-\alpha_1+\gamma_1)} \tag{11}$$

$$\times \; [2\alpha_1 - 2\gamma_1 + m(1-\alpha_1+\gamma_1)][1+\alpha_1-\gamma_1+m(1-\alpha_1+\gamma_1)]$$

$$\mathrm{E}\{N_{0,0}\} = \frac{\alpha_1-\alpha_0+\alpha_0\,\mathrm{E}\{N_{0,1}\}+\alpha_0(\alpha_1-\gamma_1)\,\mathrm{E}\{N_{0,-1}\}}{\alpha_0(1+\alpha_1-\gamma_1)} \tag{12}$$

(details are given in the Appendix). The numerical values for (10)–(12) are similar except if $|\rho|$ is close to one and/or the value of the average run length is small. In these cases the unconditional average run length, the most interesting quantity for application, can be calculated, e.g., by weighting the conditional average run lengths by the steady state probabilities α_ρ, β_ρ, and γ_ρ of the Markov chain $\{Z_n\}$. These are given by

$$\begin{aligned} \alpha_\rho &= \frac{1}{\Delta}[\alpha_0\gamma_1 + (1-\alpha_1)\alpha_0] \\ \beta_\rho &= \frac{1}{\Delta}[(1-\alpha_1)^2 - \gamma_1^2] \\ \gamma_\rho &= \frac{1}{\Delta}[\alpha_0(1-\alpha_1) + \alpha_0\gamma_1] \end{aligned} \tag{13}$$

where $\Delta = \alpha_0(1+\gamma_1-\alpha_1)+\gamma_1(\alpha_0-\gamma_1)+1+\alpha_0-\alpha_1-\alpha_1(1+\alpha_0-\alpha_1)$. In the special case where $\rho = 0$ we have $\alpha_0 = \alpha_1 = \gamma_1 = \alpha$ and

$$\mathrm{E}\{N_{0,i}\} = \frac{m(m+1)}{2\alpha}. \tag{14}$$

This result has already been mentioned by Munford (1980).

2.4 Numerical illustration

We assume that the control variable follows the normal distribution. Corresponding values for the bivariate normal distribution can be obtained according to Owen (1956). For a set of values for k and m we can calculate the run length distribution and particularly the average run length ARL(ρ) as a function of ρ. We denote by ARL(ρ) the unconditional average run length given ρ; it is obtained by weighting the conditional average run lengths (10)–(12) with the steady state probabilities α_ρ, β_ρ, and γ_ρ. *Figure 1* shows the ratio

$$\frac{\mathrm{ARL}(\rho)}{\mathrm{ARL}(0)} \tag{15}$$

as a function of ρ for various combinations of k and m so that $\mathrm{ARL}(0) = 500$. Of course, k decreases with increasing m. For $\rho > 0$ the ratios are less than one and do not deviate very much for varying k and m: The smallest ratio is that for the Shewhart chart ($m = 1$); the ratio increases with increasing m, and the largest is about three times that of the Shewhart chart. Considerable differences can be seen for $\rho < 0$. For the Shewhart chart ($m = 1$) the ratio is less than one, and the ratio again increases with increasing m. For $m = 6$ ($k \geq 1.728$), the ratio is close to one throughout the range $\rho < 0$.

insert Figure 1

Similar effects were obtained for $\mathrm{ARL}(0) = 100$; the ratios vary less, particularly for $\rho > 0$, and the chart for $m = 3$ is the one that has a ratio close to one for all $\rho < 0$.

Figure 1 shows that we easily can find values m and k so that the average run length is nearly that for $\rho = 0$ if only negative values for ρ are suspected. For positive ρ a correction similar to that suggested by Johnson and Bagshaw (1974) is necessary if a too high risk for spurious out-of-control decision should be avoided.

From (14) follows that in the uncorrelated case a particular value of $E\{N_{0,i}\}$ can be achieved by various combinations of m and α. However, the control limit m cannot be arbitrarily large. In the case of a symmetric distribution $\alpha \leq 0.5$. The parameterization with maximal m is given by

$$\begin{aligned} m^* &= \frac{1}{2}\left[\sqrt{1 + 4E\{N_{0,i}\}} - 1\right] \doteq \sqrt{E\{N_{0,i}\}} \\ k^* &= 0\,. \end{aligned} \tag{16}$$

This corresponds to a dichotomized control variable. For an ARL of 500 and 100 in the uncorrelated case, the respective approximate values for m^* are 21 and 9.

Figure 2 shows the probability distribution of the run length for $\mathrm{ARL}(0) = 500$ and various values of m and ρ. Again, the steady state probabilities α_ρ, β_ρ, and

γ_ρ were used to calculate the unconditional run length distribution. Whereas the distributions for various control limits m are rather similar for $\rho = 0.5$ and 0, considerable differences can be seen for $\rho = -0.5$ and -0.9, corresponding to the pattern discussed above for the average run length.

insert Figure 2

3 Comparison with the Johnson-Bagshaw Results

Johnson and Bagshaw (1974) investigate the effect of correlation on the run length distribution. They assume that the control variable follows a first order autoregressive or moving average process. They show that the CUSUM statistics T_n can under general conditions be approximated (for large n and consequently for large values of the ARL) by a Wiener process, and they use the distribution of the first passage time of the absolute value of the Wiener process as an approximation for the run length distribution.

The weak convergence of the CUSUM statistics to the Wiener process advocated by Johnson and Bagshaw is formally applicable to a wide class of control variable distributions, among them that of the discretized statistic (4). For the CUSUM-technique where the control variable follows a first order autoregressive process, Johnson and Bagshaw (1974) found that

$$\frac{1-\rho}{1+\rho} \tag{17}$$

is an approximate expression for the asymptotic [for large values of ARL(0)] ratio (15). In contrast to our (exact) results this ratio is independent of the control limit.

In *Figure 1*, the ratio (15) is shown together with our exact results. It turns out that the Johnson-Bagshaw ratio is acceptable for $\rho > 0$. However, for $\rho < 0$ the ratio is considerably overestimated, particularly for small values of m. The best agreement between the Johnson-Bagshaw and our results is found for m close to m^*. For the CUSUM-technique where the control variable is normally distributed and follows the first order autoregressive process (1) Monte Carlo-estimates of the ratio (15) are shown in *Figure 3* for ARL(0) = 100 and 500. For ARL(0) = 500 the ratio shows a good agreement with (15) for positive ρ, but for $\rho < 0$ the Johnson-Bagshaw ratio overestimates the actual values. It should be noted that the decrease for ρ close to -1 is not explained by (15).

insert Figure 3

An interesting question is how the ratio (15) is affected if $m = m^*$ and $ARL(0)$ becomes large. In the case of a symmetrically distributed control variable, $k = 0$, $\alpha_0 = 0.5$, and [see Johnson and Kotz (1972), p. 95] $\alpha_1 = \frac{1}{2} + \frac{1}{\pi} \arcsin \rho$. The limits of the ratios of (10)–(12) over (14) for $m^* \to \infty$ exist and coincide. We obtain for the limit

$$\frac{1 - \frac{2}{\pi} \arcsin \rho}{1 + \frac{2}{\pi} \arcsin \rho}. \tag{18}$$

This ratio more or less coincides with the exact ratio for $m = 20$ shown in *Figure 1.*

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Appendix

We recall that if the variance of Y_0 is given by (7), the conditional probabilities (6) of the process Z_i are independent of n. To derive (10)–(12) we proceed as follows: Let $x_a = E\{N_{a,1}\}$, $y_a = E\{N_{a,0}\}$ and $z_a = E\{N_{a,-1}\}$. It is not difficult to verify that the functions x_a, y_a, and z_a satisfy the set of recurrence relations

$$\begin{aligned} x_a &= 1 + \alpha_1 x_{a+1} + \beta_1 y_a + \gamma_1 z_{a-1} \\ y_a &= 1 + \alpha_0 x_{a+1} + \beta_0 y_a + \gamma_0 z_{a-1} \\ z_a &= 1 + \alpha_{-1} x_{a+1} + \beta_{-1} y_a + \gamma_{-1} z_{a-1} \end{aligned} \qquad (19)$$

for $0 \le a < m$, subject to the boundary conditions $x_m = y_m = z_m = 0$ and

$$\begin{aligned} x_0 &= 1 + \alpha_1 x_1 + \beta_1 y_0 + \gamma_1 z_0 \\ y_0 &= 1 + \alpha_0 x_1 + \beta_0 y_0 + \gamma_0 z_0 \\ z_0 &= 1 + \alpha_{-1} x_1 + \beta_{-1} y_0 + \gamma_{-1} z_0 \,. \end{aligned} \qquad (20)$$

To solve system (19), we define generating functions $H_1 = \sum_{i=0}^{m-1} s^i x_i$, $H_0 = \sum_{i=0}^{m-1} s^i y_i$ and $H_{-1} = \sum_{i=0}^{m-1} s^i z_i$. Then system (19) can be written in terms of the functions H_i as

$$\begin{aligned} (1 - \frac{\alpha_1}{s})H_1 - \beta_1 H_0 - \gamma_1 s H_{-1} &= \frac{1-s^m}{1-s} - \frac{\alpha_1}{s} x_0 + \gamma_1 z_0 - \gamma_1 s^m z_{m-1} \\ -\frac{\alpha_0}{s} H_1 + (1-\beta_0) H_0 - \gamma_0 s H_{-1} &= \frac{1-s^m}{1-s} - \frac{\alpha_0}{s} x_0 + \gamma_0 z_0 - \gamma_0 s^m z_{m-1} \end{aligned} \qquad (21)$$

$$-\frac{\alpha_{-1}}{s}H_1 - \beta_{-1}H_0 + (1-\gamma_{-1}s)H_{-1} = \frac{1-s^m}{1-s} - \frac{\alpha_{-1}}{s}x_0 + \gamma_{-1}z_0 - \gamma_{-1}s^m z_{m-1}.$$

In the symmetric case, i.e., when $\gamma_{-1} = \alpha_1$, $\gamma_0 = \alpha_0$, and $\alpha_{-1} = \gamma_1$, the solution of (21) is

$$\begin{aligned} H_1 &= \frac{1}{D}\Big[\alpha_0(1+\alpha_1-\gamma_1)x_0 + s^{m+1}(1+2\alpha_0-\alpha_1-\gamma_1+\alpha_0(1-\alpha_1+\gamma_1)z_{m-1}) \\ &- s(1+2\alpha_0-\alpha_1-\gamma_1+\alpha_0(1+3\alpha_1-3\gamma_1)x_0+\alpha_0(1-\alpha_1+\gamma_1)z_0) \\ &- s^{m+2}(\alpha_1+2\alpha_0\alpha_1-\alpha_1^2-\gamma_1-2\alpha_0\gamma_1+\gamma_1^2+\alpha_0(1-\alpha_1+\gamma_1)z_{m-1}) \\ &+ s^2(\alpha_1+2\alpha_0\alpha_1-\alpha_1^2-\gamma_1-2\alpha_0\gamma_1+\gamma_1^2+2\alpha_0(\alpha_1-\gamma_1)x_0+ \\ &\quad +\alpha_0(1-\alpha_1+\gamma_1)z_0)\Big] \end{aligned} \tag{22}$$

$$\begin{aligned} H_0 &= \frac{1}{D}\Big[\alpha_1-\alpha_0+\alpha_0x_0+\alpha_0(\alpha_1-\gamma_1)z_0+s^{m+2}(\alpha_0-\alpha_1-\alpha_0z_{m-1}) \\ &+ s^m(\alpha_0-\alpha_1-\alpha_0(\alpha_1-\gamma_1)z_{m-1}+s^2(\alpha_1-\alpha_0+\alpha_0z_0+\alpha_0(\alpha_1-\gamma_1)x_0) \\ &+ s^{m+1}(1-2\alpha_0\alpha_1+\alpha_1^2+2\alpha_0\gamma_1-\gamma_1^2+\alpha_0(1+\alpha_1-\gamma_1)z_{m-1})) \\ &- s(1-2\alpha_0\alpha_1+\alpha_1^2+2\alpha_0\gamma_1-\gamma_1^2+\alpha_0(1+\alpha_1-\gamma_1)x_0+ \\ &\quad +\alpha_0(1+\alpha_1-\gamma_1)z_0)\Big] \end{aligned} \tag{23}$$

and

$$\begin{aligned} H_{-1} &= \frac{1}{D}\Big[\alpha_1+2\alpha_0\alpha_1-\alpha_1^2-\gamma_1-2\alpha_0\gamma_1+\gamma_1^2+2\alpha_0(\alpha_1-\gamma_1)z_0 \\ &- \alpha_0(1+\alpha_1-\gamma_1)s^{m+2}z_{m-1}+\alpha_0(1-\alpha_1+\gamma_1)x_0+\alpha_0(1+\alpha_1-\gamma_1)s^2z_0 \\ &+ s^{m+1}(1+2\alpha_0-\alpha_1-\gamma_1+\alpha_0(1+3\alpha_1-3\gamma_1)z_{m-1}) \\ &- s(1+2\alpha_0-\alpha_1-\gamma_1+\alpha_0(1+3\alpha_1-3\gamma_1)z_0+\alpha_0(1-\alpha_1+\gamma_1)x_0) \\ &+ s^m(\alpha_1^2-\alpha_1-2\alpha_0\alpha_1+\gamma_1+2\alpha_0\gamma_1-\gamma_1^2-2\alpha_0(\alpha_1-\gamma_1)z_{m-1})\Big], \end{aligned} \tag{24}$$

where $D = \alpha_0(1-s)^3(1+\alpha_1-\gamma_1)$. The generating functions H_i contain the unknown functions x_0, z_0, and z_{m-1}. In order to determine these unknowns, we observe, that the H_i have a pole of order 3 at $s = 1$. However, the H_i are polynomials by definition, thus this singularity must be removable. So we may take any of the functions H_i and consider its limit as $s \to 1$. In particular, if we use H_{-1}, then upon differentiating the numerator once and passing to the limit $s \to 1$, we obtain a first equation

$$\alpha_0(x_0 - z_0 + z_{m-1}) = m(1+2\alpha_0-\alpha_1-\gamma_1). \tag{25}$$

Differentiating the numerator of H_{-1} twice with respect to s and forming again the limit $s \to 1$ yields the equation

$$\begin{aligned} 2\alpha_0(1+\alpha_1-\gamma_1)z_0 - 2\alpha_0[1+\alpha_1-\gamma_1+m(1-\alpha_1+\gamma_1)]z_{m-1} = \\ = \quad -m(1+2\alpha_0+2\alpha_0\alpha_1-\alpha_1^2-2\gamma_1-2\alpha_0\gamma_1+\gamma_1^2) \\ -m^2(1+2\alpha_0-2\alpha_1-2\alpha_0\alpha_1+\alpha_1^2+2\alpha_0\gamma_1-\gamma_1^2)\,. \end{aligned} \tag{26}$$

A third equation may be obtained by observing that

$$z_0 = \lim_{s\to 0} H_{-1},$$

which leads to

$$\alpha_0(1-\alpha_1+\gamma_1)x_0 - \alpha_0(1-\alpha_1+\gamma_1)z_0 = \gamma_1(1+2\alpha_0-\gamma_1)-\alpha_1(1+2\alpha_0-\alpha_1). \tag{27}$$

Solving (25),(26), and (27) for the unknowns x_0, z_0, and z_{m-1} yields

$$x_0 = \frac{m(1+2\alpha_0-\alpha_1-\gamma_1)[1+3\alpha_1-3\gamma_1+m(1-\alpha_1+\gamma_1)]}{2\alpha_0(1+\alpha_1-\gamma_1)} \tag{28}$$

$$\begin{aligned} z_0 &= \frac{(1+2\alpha_0-\alpha_1-\gamma_1)}{2\alpha_0(1+\alpha_1-\gamma_1)(1-\alpha_1+\gamma_1)} \\ &\times [2\alpha_1-2\gamma_1+m(1-\alpha_1+\gamma_1)][1+\alpha_1-\gamma_1+m(1-\alpha_1+\gamma_1)] \end{aligned} \tag{29}$$

$$z_{m-1} = \frac{(1+2\alpha_0-\alpha_1-\gamma_1)[\alpha_1-\gamma_1+m(1-\alpha_1+\gamma_1)]}{\alpha_0(1-\alpha_1+\gamma_1)} \tag{30}$$

Finally, y_0 remains to be determined. Again we note that $y_0 = \lim_{s\to 0} H_0$, which leads to the equation

$$y_0 = \frac{\alpha_1-\alpha_0+\alpha_0 x_0+\alpha_0(\alpha_1-\gamma_1)z_0}{\alpha_0(1+\alpha_1-\gamma_1)}\,. \tag{31}$$

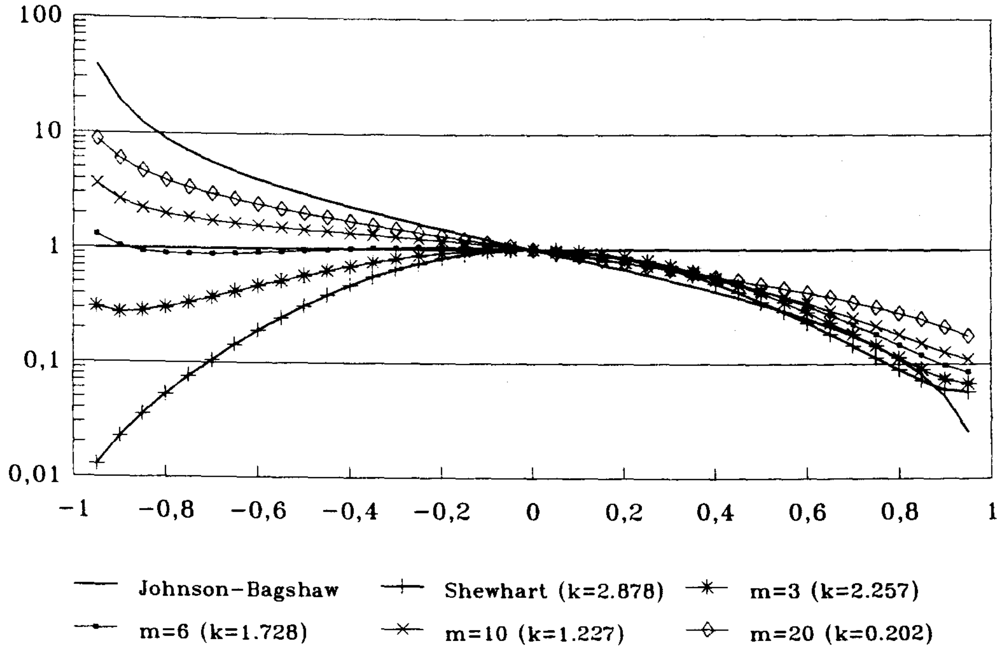

Figure 1: **Ratio ARL(ρ)/ ARL(0) by ρ for the discretized CUSUM-technique where the control variable follows the first order autoregressive process (1) [ARL(0) = 500], and Johnson-Bagshaw ratio (17).**

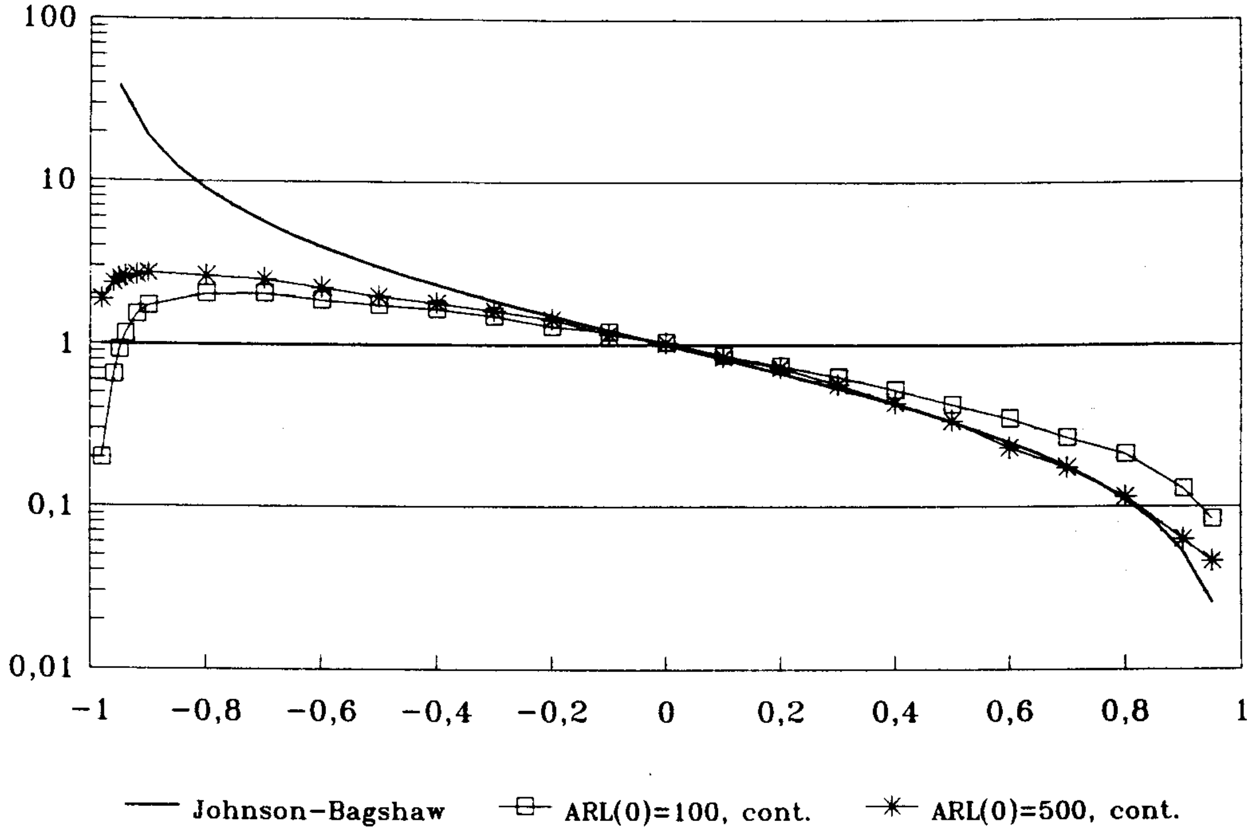

Figure 2: Run length distribution of the ratio ARL(ρ)/ ARL(0) by ρ for the discretized CUSUM-technique where the control variable follows the first order autoregressive process (1) [ARL(0) = 500].

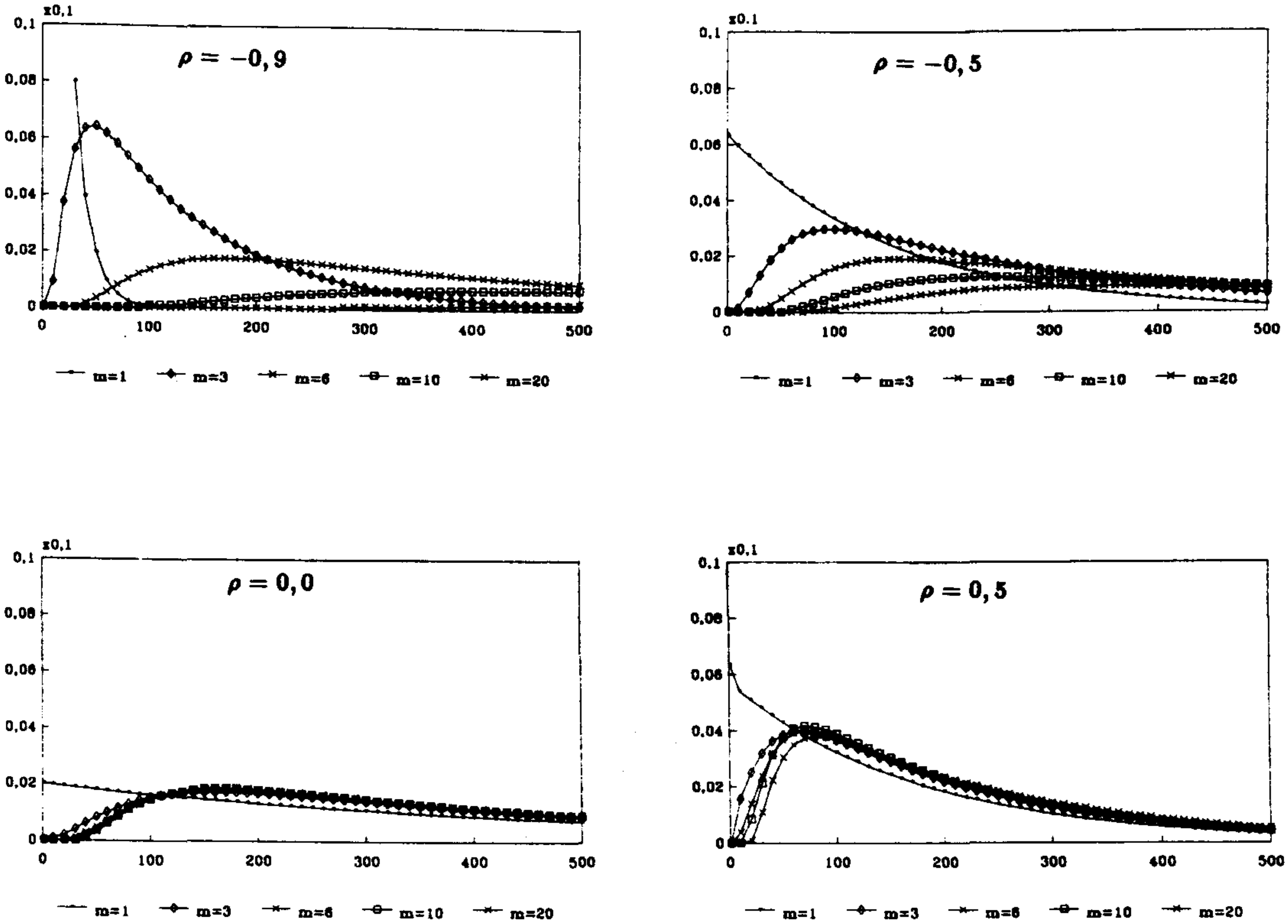

Figure 3: Monte Carlo-estimates of the ratio ARL(ρ)/ ARL(0) by ρ for the CUSUM-technique where the control variable is normally distributed and follows the first order autoregressive process (1), and Johnson-Bagshaw ratio (17).