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DOI:
10.57938/da1bab12-bab7-48c9-b345-2fa7d8ca3361

Published: 01/01/1999

Document Version
Publisher's PDF, also known as Version of record

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Forschungsberichte

Bericht 66
April 1999

http://statmath.wu-wien.ac.at/

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ABSTRACT. In this paper we consider an analogue of the classical simple random walk on the set of integers which has correlated increments. In particular we are interested in the distribution of absorption times and the maximum of such processes.

Accepted for publication in *The Journal of Statistical Planning and Inference*, April 27, 1999.

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ABSTRACT. In this paper we consider an analogue of the classical simple random walk on the set of integers which has correlated increments. In particular we are interested in the distribution of absorption times and the maximum of such processes.

1. Introduction

Let us consider a random walk process $S_n = X_1 + X_2 + \ldots + X_n$, where the increments X_i take their values in the set $\mathcal{S} = \{-1, +1\}$ and are governed by a two-state Markov chain with state space $\mathcal{S}$ and one-step transition matrix

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}.$$

Thus

$$P(X_i = 1 | X_{i-1} = 1) = P(X_i = -1 | X_{i-1} = -1) = \alpha$$
$$P(X_i = 1 | X_{i-1} = -1) = P(X_i = -1 | X_{i-1} = 1) = 1 - \alpha$$

Throughout this paper we assume that the parameter α satisfies $0 < \alpha < 1$. This assumption avoids the following trivial extreme cases: if $\alpha = 1$, then $S_n = n$ or $S_n = -n$ depending on the value of X_0. In the case $\alpha = 0$ the governing Markov chain is ultimately periodic and S_n oscillates between -1 and 0 or 0 and $+1$, again depending on the value of X_0. Note that for $\alpha = 1/2$, S_n is just the ordinary simple random walk. A physical interpretation of this type of random walk is the following: consider a particle moving on the one-dimensional integer lattice. At each step the particle keeps moving into the same direction with probability α, and it changes its direction with probability $1 - \alpha$. The persistent random walk is thus a generalization of the classical simple random walk, allowing for correlated increments.

The process S_n is usually referred to as *correlated random walk*. The special case with symmetric transition matrix of the governing chain we are interested in this paper is also called *persistent random walk*. It plays an important role in statistical physics, where it is used as a model for diffusion in turbulent media (Gillis J. (1955), Weiss G.H. (1994), pp. 33). For more information about correlated walks the reader may consult the paper of Lal R., Bhat U.N. (1989) and the references cited there.

Our analysis of S_n will be based almost exclusively on generating functions, an approach which seems to be a reasonable alternative to recent attempts in lattice paths combinatorics which provide insight into the structure of S_n by counting paths with respect to turns. This is accomplished by an ingenuine technique due to Krattenthaler (see e.g. Krattenthaler Ch. (1997)) which is based on a representation of the simple lattice paths originating from S_n in terms of two-rowed integer arrays. Generating functions, however, offer a substantial methodological advantage, because they provide us directly with important asymptotic information about the process S_n.

The plan of the paper is as follows: In the next section we consider the distribution of the time to absorption at a boundary. Explicit formulae are obtained by means of multivariate Lagrange inversion. The corresponding generating functions yield immediately the asymptotic distribution of the maximum in section 3. Finally we obtain an asymptotic formula for the expectation of the maximum by means of singularity analysis.

2. The time to absorption

For positive integers a we define the random stopping time

$$T_\pm^a = \inf\{n : S_n = a\}$$

with probability function

$$g_\pm(n, a) = P(T_a^\pm = n | X_0 = \pm 1)$$

and generating function

$$G_\pm(s, a) \equiv G_\pm(a) = \sum_{n \geq 0} s^n g_\pm(n, a).$$

These generating functions satisfy the boundary condition $G_\pm(0) = 1$ and

$$G_+(a) = G^+(1)^a \tag{1}$$

$$G_-(a) = G_-(1) G_+(1)^{a-1} \tag{2}$$

The latter equations follow from a simple argument: a first passage through $a > 1$ implies first passages through $1, 2, \dots, a-1$, each one terminated by a $+1$-step, which in turn is the zero-th step of the path segment immediately after. Let us put $G_- = G_-(1)$ and $G_+ = G_+(1)$ for notational convenience.

By conditioning on the first step, we find that the functions $G_\pm(a)$ also satisfy the following system of difference equations:

$$G_-(a) = s(1-\alpha)G_+(a-1) + s\alpha G_-(a+1) \tag{3}$$
$$G_+(a) = s\alpha G_+(a-1) + s(1-\alpha)G_-(a+1)$$

which together with (1) and (2) yield:

$$G_- = s(1-\alpha) + s\alpha G_- G_+ \tag{4}$$
$$G_+ = s\alpha + s(1-\alpha)G_- G_+$$

From (4) it follows that the functions G_- and G_+ are solutions of the quadratic equations

$$s\alpha G_+^2 - G_+(1+s^2(2\alpha-1)) + s\alpha = 0 \tag{5}$$
$$s(1-\alpha)G_-^2 - G_-(1-s^2(2\alpha-1)) + s(1-\alpha) = 0 \tag{6}$$

The admissible roots are (we require these functions to vanish at $s=0$):

$$G_+ = \frac{1+s^2(2\alpha-1) - \sqrt{(1+s^2(2\alpha-1))^2 - 4s^2\alpha^2}}{2\alpha s} \tag{7}$$
$$G_- = \frac{1-s^2(2\alpha-1) - \sqrt{(1-s^2(2\alpha-1))^2 - 4s^2(1-\alpha)^2}}{2(1-\alpha)s} \tag{8}$$

The square roots have to be taken in such a way that their real part is positive.

It is of course possible to obtain a power series expansion of these functions in a direct manner, however, a more elegant way to proceed is to apply the multivariate generalization of the Lagrange inversion theorem to the system (4).

Theorem 2.1. *For* $a \geq 1$*:*

(9)
$$g_+(2n+a,a) = \sum_{i\geq 0} \alpha^{a+2i}(1-\alpha)^{2n-2i}\binom{n-1}{i}\left[\binom{n+a-1}{n-i} - \frac{\alpha^2}{(1-\alpha)^2}\binom{n+a-1}{n-2-i}\right],$$

(10)
$$g_-(2n+a,a) = \sum_{i\geq 0} \alpha^{a+2i-1}(1-\alpha)^{2n-2i+1}\binom{n}{i}\left[\binom{n+a-2}{n-i} - \frac{\alpha^2}{(1-\alpha)^2}\binom{n+a-2}{n-2-i}\right]$$

Proof. We apply multivariate Lagrange inversion (see Goulden and Jackson (1983), pp. 21, Henrici (1988), pp. 100) to the system (4), viz:

$$G_- = s[1-\alpha+\alpha G_- G_+]$$
$$G_+ = s[\alpha + (1-\alpha)G_- G_+]$$

Put $\lambda_1 = G_-(a), \lambda_2 = G_+(a)$ and $f(\lambda_1, \lambda_2) = \lambda_2^a$. Furthermore put

$$\Phi_1 = \Phi_1(\lambda_1, \lambda_2) = 1 - \alpha + \alpha\lambda_1\lambda_2$$
$$\Phi_2 = \Phi_2(\lambda_1, \lambda_2) = \alpha + (1-\alpha)\lambda_1\lambda_2,$$

and suppose that

$$w_1 = s\Phi_1(w_1, w_2)$$
$$w_2 = s\Phi_2(w_1, w_2)$$

for arbitrary indeterminates w_1, w_2.

Then by multivariate Lagrange inversion:

$$f(w_1(s), w_2(s)) = \sum_{i,j} s^{i+j}[\lambda_1^i\lambda_2^j]f(\lambda_1,\lambda_2)\Phi_1^i\Phi_2^j \det\left\|\delta_{xy} - \frac{\lambda_y}{\Phi_x}\frac{\partial\Phi_x}{\partial\lambda_y}\right\|, \quad x, y = 1, 2$$

δ_{xy} denotes the Kronecker delta and $[\lambda_1^i\lambda_2^j]$ is the bivariate coefficient operator. The determinant equals

$$1 - \frac{\alpha\lambda_1\lambda_2}{\Phi_1} - \frac{(1-\alpha)\lambda_1\lambda_2}{\Phi_2}.$$

Thus

$$G_+(a, s) = \sum_{i,j} s^{i+j}[\lambda_1^i\lambda_2^j]\lambda_2^a\Phi_1^i\Phi_2^j(1 - \frac{\alpha\lambda_1\lambda_2}{\Phi_1} - \frac{(1-\alpha)\lambda_1\lambda_2}{\Phi_2})$$
$$= \alpha(1-\alpha)\sum_{i,j} s^{i+j}[\lambda_1^i\lambda_2^{j-a}]\Phi_1^{i-1}\Phi_2^{j-1}(1 - \lambda_1^2\lambda_2^2).$$

Now set $i + j = 2n + a$ and expand $\Phi_1^{i-1}\Phi_2^{j-1}$ according to the binomial theorem. As a result we get

$$g_+(2n+a, a) = \sum_{i\geq 0}\binom{n-1}{i}\binom{n+a-1}{n-i}\alpha^{a+2i}(1-\alpha)^{2n-2i}$$
$$-\sum_{i\geq 0}\binom{n-1}{i}\binom{n+a-1}{n-2-i}\alpha^{a+2+2i}(1-\alpha)^{2n-2-2i}.$$

This proves (9).

Similarly, put $f(\lambda_1, \lambda_2) = \lambda_1\lambda_2^{a-1}$. Proceeding just the same way we get (10). □

It should be noted that the probabilities $g_\pm(n, a)$ may be obtained also directly by a purely combinatorial argument (Mohanty (1979), p. 135)).

3. The distribution of the maximum

The functions $g_\pm(n, a)$ and their corresponding generating functions determined in the previous section play a prominent role in connection with the maximum statistics of the process S_n. This is easy to see, because, if we let

$$M_n^\pm = \max_{0\le i\le n} S_i,$$

(the superscript again indicating the conditioning on the zero-th step), then the following identity holds

$$(11)\qquad \begin{aligned} P(M_n^\pm \ge a) &= P(T_a^\pm \le n) \\ &= \sum_{i\le n} g_\pm(i, a). \end{aligned}$$

Thus in principle the distribution of the maximum is known and given in essence by a summation of the functions (9) and (10) respectively.

Let us now determine the asymptotic distribution of $M_n^\pm$.

Theorem 3.1. *Let $x > 0$ be fixed. Then as $n \to \infty$*

$$(12)\qquad P(M_n^\pm = x\sqrt{n}) \sim \sqrt{\frac{1-\alpha}{\alpha}}\sqrt{\frac{2}{n\pi}}\exp\left[-x^2\frac{1-\alpha}{2\alpha}\right].$$

Proof. From the identity (11) and the fact that

$$P(M_n^\pm = a) = P(M_n^\pm \ge a) - P(M_n^\pm \ge a+1),$$

it follows that

$$P(M_n^\pm = a) = [s^n]\frac{G_\pm(a)(1-G_\pm)}{1-s}.$$

Consider first the case M_n^+. By Cauchy's formula and the representation (7):

$$\begin{aligned} P(M_n^+ = a) =& \frac{1}{2\pi i}\oint\left(\frac{1+s^2(2\alpha-1)-\sqrt{(1+s^2(2\alpha-1))^2-4\alpha^2s^2}}{2\alpha s}\right)^a \times \\ &\times\left(\frac{2\alpha s-1-s^2(2\alpha-1)+\sqrt{(1+s^2(2\alpha-1))^2-4\alpha^2s^2}}{2\alpha s}\right)\frac{s^{-n-1}}{1-s}\,ds \end{aligned}$$

As contour of integration we choose

$$(13)\qquad s = Re^{\theta i}, \qquad |R| < 1, \qquad -\pi \le \theta \le \pi.$$

Now put $s = e^{-\nu/n}$. This substitution maps the contour (13) to the new contour

$$C : \nu = -n \log R - n\theta i, \tag{14}$$

which is a straight line segment parallel to the imaginary axis oriented from top to bottom, lying in the half-plane $\Re(\nu) > 0$. If we change the orientation from bottom to top, the we get

$$P(M_n^+ = a) = \frac{1}{2\pi i}\int_C \left(\frac{1 + (2\alpha-1)e^{-2\nu/n} - \sqrt{(1+(2\alpha-1)e^{-2\nu/n})^2 - 4\alpha^2 e^{-2\nu/n}}}{2\alpha e^{-\nu/n}}\right)^a \times$$
$$\times \left(\frac{2\alpha e^{-\nu/n} - 1 - e^{-2\nu/n}(2\alpha-1) - \sqrt{(1+(2\alpha-1)e^{-2\nu/n})^2 - 4\alpha^2 e^{-2\nu/n}}}{2\alpha e^{-\nu/n}}\right) \times$$
$$\times \frac{e^\nu}{1-e^{-\nu/n}}\frac{d\nu}{n}.$$

Expanding the exponential functions and putting

$$a = x\sqrt{n},$$

we find that as $n \to \infty$ the first factor of the integrand is equal to

$$\exp\left[-x\sqrt{\frac{2(1-\alpha)}{\alpha}}\sqrt{\nu}\right]\left(1 + O\left(\frac{1}{n}\right)\right).$$

The second factor turns out to be

$$\sqrt{\nu}\sqrt{\frac{2(1-\alpha)}{\alpha}}\sqrt{\frac{1}{n}} + O\left(\frac{1}{n}\right).$$

Finally for the last factor, including the term $1/n$, we get:

$$\frac{1}{1-e^{-\nu/n}}\frac{1}{n} = \frac{1}{\nu} + O\left(\frac{1}{n}\right).$$

Thus

$$P(M_n^+ = x\sqrt{n}) = \frac{1}{2\pi i}\sqrt{\frac{2(1-\alpha)}{\alpha}}\sqrt{\frac{1}{n}}\int_C \frac{1}{\sqrt{\nu}}\exp\left[-x\sqrt{\frac{2(1-\alpha)}{\alpha}}\sqrt{\nu}\right] e^\nu \left(1 + O\left(\frac{1}{n}\right)\right) d\nu.$$

For $n \to \infty$ the integral converges to the complex Laplace inversion integral of the transform (note that the contour (14) is just the principal part of a Bromwich contour):

$$\frac{1}{\sqrt{\nu}}\exp\left[-x\sqrt{\frac{2(1-\alpha)}{\alpha}}\sqrt{\nu}\right]$$

The inverse of this transform is well known and therefore we have, as $n \to \infty$:

$$P(M_n^+ = x\sqrt{n}) \sim \sqrt{\frac{2(1-\alpha)}{\alpha}}\frac{1}{\sqrt{n\pi}}\exp\left[-x^2\frac{1-\alpha}{2\alpha}\right].$$

This, however, is the same as (12). Following the same line of arguments it can be shown that for $n \to \infty$ M_n^- has the same distribution as M_n^+. □

4. The expectation of the maximum

Since $M_n^\pm$ is a nonnegative random variable we get its expectation by summation of the tail probabilities of its distribution. In terms of generating functions

$$\sum s^n P(M_n^\pm \geq a) = G_\pm(a)\frac{1}{1-s},$$

and because

$$G_+(a) = G_+^a, \qquad G_-(a) = G_- G_+^{a-1},$$

we have

$$E(M_n^+) = [s^n]\frac{G_+}{1-G_+}\frac{1}{1-s} \tag{15}$$

and

$$E(M_n^-) = [s^n]\frac{G_-}{1-G_+}\frac{1}{1-s}. \tag{16}$$

We shall prove now:

Theorem 4.1. *As* $n \to \infty$

$$E(M_n^+) = -\frac{1}{2} + \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{2n}{\pi}} + O\left(\frac{1}{\sqrt{n}}\right). \tag{17}$$

and

$$E(M_n^-) = \frac{1}{2} + \frac{\alpha}{1-\alpha} + \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{2n}{\pi}} + O\left(\frac{1}{\sqrt{n}}\right). \tag{18}$$

Proof: Let us consider first (15). By (7) and (8), we can write G_+ as

$$G_+ = \frac{1 + s^2(2\alpha - 1) - \sqrt{(1-s^2)(1-s^2(2\alpha-1)^2)}}{2\alpha s}$$

and

$$G_- = \frac{1 - s^2(2\alpha - 1) - \sqrt{(1-s^2)(1-s^2(2\alpha-1)^2)}}{2(1-\alpha)s}$$

By some algebra we find that

$$\frac{G_+}{1-G_+}\frac{1}{1-s} = \frac{1}{2(1-s)}\frac{(s-1)(s(2\alpha-1)-1)-\sqrt{(1-s^2)(1-s^2(2\alpha-1)^2)}}{(1-s)(s(2\alpha-1)-1)}$$
$$= -\frac{1}{2(1-s)} + A(s),$$

where

$$A(s) = \frac{1}{2}(1+s)^{1/2}(1-s)^{-3/2}(1+s(2\alpha-1))^{1/2}(1-s(2\alpha-1))^{-1/2}.$$

Of course

$$[s^n]\left(-\frac{1}{2(1-s)}\right) = -\frac{1}{2}.$$

For $A(s)$ we note that it has singularities at ± 1 and $\pm\frac{1}{2\alpha-1}$. Only the singularities at ± 1 have to be taken into account, because the others do not lie in or on the unit circle, since $0 < \alpha < 1$ by assumption.

Expanding $A(s)$ around $s = 1$, we get

$$A(s) = \frac{\sqrt{2}}{2}\sqrt{\frac{\alpha}{1-\alpha}}(1-s)^{-3/2} + O((1-s)^{-1/2}). \tag{19}$$

If we expand $A(s)$ around $s = -1$, we have

$$A(s) = O((1+s)^{1/2}). \tag{20}$$

Now

$$[s^n](1-s)^{-a} = \frac{n^{a-1}}{\Gamma(a)}\left(1 + O(\frac{1}{n})\right). \tag{21}$$

It follow that the singularity at -1 contributes only terms of order $O(n^{-3/2})$ and below. Furthermore

$$[s^n](1-s)^{-1/2} = O(n^{-1/2}).$$

Therefore only the first term in (19) contributes essentially to $E(M_n^+)$, and finally we get (17) using (21).

Similarly

$$\frac{G_-}{1-G_+}\frac{1}{1-s} = -\frac{1}{2(1-\alpha)}\frac{\alpha+s(2\alpha-1)}{1-s} + B(s),$$

where

$$B(s) = \frac{\alpha - s(2\alpha-1)}{2(1-\alpha)}(1-s)^{-3/2}(1+s)^{1/2}(1-s(2\alpha-1))^{-1/2}(1+s(2\alpha-1))^{1/2}.$$

Again, expanding $B(s)$ around $s = 1$ yields

$$B(s) = \frac{\sqrt{2}}{2}\sqrt{\frac{\alpha}{1-\alpha}}(1-s)^{-3/2} + O((1-s)^{-1/2}).$$

This is the same as the expansion of $A(s)$. For the constant term we get

$$[s^n]\left(-\frac{1}{2(1-\alpha)}\frac{\alpha + s(2\alpha-1)}{1-s}\right) = \frac{1-3\alpha}{2(1-\alpha)} = \frac{1}{2} - \frac{\alpha}{1-\alpha}.$$

□.

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