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An operator limit

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In our work on weak convergence of random walks with increments governed by a finite state Markov chain (also called correlated random walk), we encountered the following problem:

Let $\mathcal{X}$ be a Banach space and $\mathcal{B}(\mathcal{X})$ the set of all bounded linear operators on $\mathcal{X}$ to itself. Furthermore let $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{X})$, with $\mathbf{A}^2 = \mathbf{A}$. Determine the limit

$$\lim_{n\to\infty} (\mathbf{A} + \frac{t}{n}\mathbf{B})^n,$$

for any fixed t, $0 < t < \infty$.

Solution:

$$\begin{aligned}\lim_{n\to\infty} (\mathbf{A} + \frac{t}{n}\mathbf{B})^n &= \mathbf{A}e^{t\mathbf{BA}} \\ &= e^{t\mathbf{AB}}\mathbf{A}.\end{aligned}$$

Proof: Let $\mathbf{Q}_n(k)$ denote the coefficient of t^k in the polynomial expansion of $(\mathbf{A} + \frac{t}{n}\mathbf{B})^n$ and assume that $k \geq 0$ is fixed. Then $\mathbf{Q}_n(k)$ is the sum of $\binom{n}{k}$ products of the operators $\mathbf{A}$ and $\mathbf{B}$ involving k $\mathbf{B}$'s and $n-k$ $\mathbf{A}$'s in any possible combination. Out of these products consider only those which are of the following form

$$\mathbf{A}^{i_1}\mathbf{B}\mathbf{A}^{i_2}\mathbf{B}\ldots\mathbf{B}\mathbf{A}^{i_{k+1}} = \mathbf{A}(\mathbf{BA})^k = (\mathbf{AB})^k\mathbf{A},$$

$$i_1 + i_2 + \ldots + i_{k+1} = n - k.$$

The number of these products equals the number of ways $n-k$ balls can be placed into $k+1$ cells in such a way that none of the cells remains empty. Thus it equals

$$\binom{n-k-1}{k},$$

and we assume $2k < n$. Therefore, if k is fixed, we have

$$\mathbf{Q}_n(k) = \frac{1}{n^k}\binom{n-k-1}{k}\mathbf{A}(\mathbf{BA})^k + \frac{1}{n^k}\mathbf{C}_n(k),$$

where $\mathbf{C}_n(k)$ is a sum of $\binom{n}{k} - \binom{n-k-1}{k}$ products of the operators $\mathbf{A}$ and $\mathbf{B}$, involving k $\mathbf{B}$'s. Now

$$\|\mathbf{C}_n(k)\| \leq \left[1 - \frac{(n-k-1)\dots(n-2k)}{n(n-1)\dots(n-k+1)}\right] \|\mathbf{B}\|^k,$$

and

$$\lim_{n\to\infty} \|\mathbf{C}_n(k)\| = 0.$$

As a result we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{Q}_n(k) &= \lim_{n\to\infty} \frac{1}{n^k} \frac{(n-k-1)\dots(n-2k)}{k!} \mathbf{A}(\mathbf{BA})^k + \lim_{n\to\infty} \frac{1}{n^k} \mathbf{C}_n(k) \\
&= \frac{1}{k!} \mathbf{A}(\mathbf{BA})^k \\
&= \frac{1}{k!} (\mathbf{AB})^k \mathbf{A}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} (\mathbf{A} + \frac{t}{n}\mathbf{B})^n &= \sum_{k\geq 0} \frac{t^k}{k!} (\mathbf{AB})^k \mathbf{A} = \sum_{k\geq 0} \frac{t^k}{k!} \mathbf{A}(\mathbf{BA})^k \\
&= e^{t\mathbf{AB}} \mathbf{A} = \mathbf{A} e^{t\mathbf{BA}}.
\end{aligned}$$

As in interesting consequence we have also the remarkable identity

$$e^{-\mathbf{BA}t} \mathbf{A} e^{\mathbf{BA}t} - e^{\mathbf{AB}t - \mathbf{BA}t} \mathbf{A} = \mathbf{0},$$

which has important applications in the analysis of Hamiltonian systems.