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DOI:
10.57938/ae2dd305-dbd0-4fc4-b4c6-4505655942c1

Published: 01/01/1991

Document Version
Publisher's PDF, also known as Version of record

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Forschungsberichte

Bericht 22
1991

http://statmath.wu-wien.ac.at/

A note on M/M/1 queues with gamma-distributed vacations

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Abstract

In this note we derive a simple formula for the density of $S_n = \sum_{i=1}^{n} T_i$, where T_i is the duration of the i-th busy cycle in an M/M/1 queueing model with gamma distributed server vacations. Using this result we find the distribution of the queue length by standard renewal arguments.

1 Introduction

Recently Takagi (1990) has presented a paper in which the Laplace transforms of various queueing characteristics for M/G/1 systems with server vacations are given. However, it is by no means trivial to determine the inverse transforms, even in the special case of M/M/1. In this note we discuss the problem of determining the transient distribution of the queue length in an M/M/1 queueing model where customers arrive with rate λ, are served with rate μ and the server goes an a vacation whenever the queue becomes empty. We assume that the vacations are mutually independent random variates with common density

$$w(t) = \frac{e^{-\pi t}\pi^{\nu}t^{\nu-1}}{(\nu-1)!},$$

where π and ν are arbitrary reals, satisfying $\pi > 0$ and $\nu \geq 1$. The key result is the density function of $S_n = \sum_{i=1}^{n} T_i$, where T_i is the duration of the i-th busy cycle. Rather surprisingly it will turn out, that this density of particularly simple form.

2 The busy cycle

We define a busy cycle as a vacation followed by a completed busy period. It is assumed that vacations can start only at those time points where busy periods terminate, thus at the beginning of a vacation there are no customers in the queue. A cycle may be degenerate, which will happen, if there are no arrivals during a vacation. In such a case we assume that the busy period following this vacation has length zero.

Let T_i denote the length of a cycle. Since the lengths of the server vacations are i.i.d. random variates and independent of the service process, it follows, that also the T_i are mutually independent with density function $g(t)$. Some reflection shows that under these conditions the Laplace transform (L.T.) of $g(t)$, viz. $g^\star(s)$ is given by

$$g^\star(s) = \sum_{a\geq 0} h_a^\star(s)\xi^a(s), \tag{1}$$

where $h_a^\star(s)$ is the L.T. of

$$h_a(t) = w(t)\frac{e^{-\lambda t}(\lambda t)^a}{a!},$$

and $\xi(s)$ is the L.T. of the density of a busy period initiated by a single customer. It is well known (see e.g. Prabhu (1965, p. 8)), that

$$\xi(s) = \frac{1}{2\lambda}[s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}], \tag{2}$$

where the square root has to be taken so that its real part is positive. Besides other functional relations the function $\xi(s)$ satisfies

$$\xi(s)\eta(s) = \frac{\mu}{\lambda}, \tag{3}$$

where

$$\eta(s) = \frac{1}{2\lambda}[s+\lambda+\mu+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}]. \tag{4}$$

Equation (1) can be simplified further to yield

$$g^\star(s) = w^\star(s+\lambda-\lambda\xi(s)), \tag{5}$$

a result, which may be found in Takagi (1990).

Henceforth we assume that the server vacations have a gamma density

$$w(t) = \frac{e^{-\pi t}\pi^{\nu}t^{\nu-1}}{(\nu-1)!},$$

where π is real positive and ν is any real number ≥ 1.

Let $S_n = \sum_{i=0}^{n} T_i$. The density of S_n is given by the $n-$fold convolution of $g(t)$, which we denote by $g^{(n)}(t)$. We will show that this density has the following interesting form:

$$g^{(n)}(t) = \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a\geq 0} \binom{a+\nu n-1}{a} f^{(a+\nu n)}(t)(1-\frac{\pi}{\mu})^{a}, \tag{6}$$

with

$$f^{(a+\nu n)}(t) = e^{-(\lambda+\mu)t}\rho^{-\frac{a+\nu n}{2}}\frac{a+\nu n}{t}I_{a+\nu n}(2t\sqrt{\lambda\mu}),$$

and $I_a(2t\sqrt{\lambda\mu})$ denotes the modified Bessel function of (possibly) fractional order a.

To prove (6) we note that according to (5) the L.T. of $g^{(n)}(t)$ is given by

$$g^{n*}(s) = \left(\frac{\pi}{\pi+\lambda+s-\lambda\xi(s)}\right)^{\nu n}.$$

Because of (3), this can be rewritten as

$$\begin{aligned} g^{n*}(s) &= \left(\frac{\pi}{\pi-\mu+\mu/\xi(s)}\right)^{\nu n} \\ &= \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a\geq 0} \binom{a+\nu n-1}{a} \xi^{a+\nu n}(s)(1-\frac{\pi}{\mu})^{a}. \end{aligned}$$

The result follows upon inverting $\xi^{a+\nu n}(s)$.

Some remarks are in order concerning (6). First, $g^{(n)}(t)$ turns out to be a weighted sum of densities $f^{(a+\nu n)}(t)$. These densities are proper, if $\lambda < \mu$ and equal the densities of busy periods initiated by $a+\nu n$ customers, if $a+\nu n$ is a positive integer. However, such an interpretation is in general not possible, if ν is an arbitrary real number ≥ 1. Furthermore, we note, that when $\pi < \mu$, the weights in (6) are the probabilities of a negative binomial distribution. And finally, (6) remains valid even in the special case $\pi = \mu$, where (6) simplifies to

$$g^{(n)}(t) = e^{-(\lambda+\mu)t}\rho^{-\frac{\nu n}{2}}\frac{\nu n}{t}I_{\nu n}(2t\sqrt{\lambda\mu}).$$

3 The distribution of the queue length

Formula (6) allows us to state the distribution of the queue length in comparatively simple terms. Let $Q(t)$ denote the number of customers in the queue at time t and define an indicator function $\sigma(t)$ by

$$\sigma(t) = \begin{cases} 1 & \text{if the server is busy at time } t \\ 0 & \text{otherwise} \end{cases}$$

Furthermore let $r_j(t)$ denote the probability that at time t the server is busy for the first time with j customers waiting, provided at time zero a vacation started. Then we have

$$r_j(t) = \sum_{i \geq 1} h_i(t) * {}^0P_{ij}(t),$$

where $*$ denotes convolution and

$${}^0P_{ij}(t) = P(Q(t) = j, Q(s) > 0, 0 < s < t | Q(0) = j).$$

The functions ${}^0P_{ij}(t)$ are well known and given for instance in Prabhu (1965, p. 15):

$${}^0P_{ij}(t) = e^{-(\lambda+\mu)t} \rho^{\frac{j-i}{2}} \left[I_{|j-i|}(2t\sqrt{\lambda\mu}) - I_{i+j}(2t\sqrt{\lambda\mu}) \right]$$

Let us now assume that the server is busy at time zero with m customers waiting. Then the event $\{Q(t) = j, \sigma(t) = 1\}$ implies that the server had $n \geq 0$ vacations in $(0, t)$, where the last vacation terminated at some time $0 < s < t$ with a positive number of customers having arrived during that vacation. Furthermore the server is continuously busy in (s, t). Hence we have

$$P(Q(t) = j, \sigma(t) = 1 | Q(0) = m, \sigma(0) = 1) =$$

$$= \quad {}^0P_{mj}(t) + f^{(m)}(t) * r_j(t) * \sum_{n \geq 0} g^{(n)}(t).$$

This formula may be simplified further by observing that

$$f^{(m)}(t) * g^{(n)}(t) = \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a \geq 0} \binom{a + \nu n - 1}{a} f^{(a+m+\nu n)}(t)(1 - \frac{\pi}{\mu})^a,$$

and

$$f^{(a+m+\nu n)}(t) * {}^0P_{ij}(t) =$$
$$= \quad e^{-(\lambda+\mu)t} \rho^{\frac{j-i-a-\nu n-m}{2}} \left[I_{a+\nu n+m+|j-i|}(2t\sqrt{\lambda\mu}) - I_{a+\nu n+m+j+i}(2t\sqrt{\lambda\mu}) \right].$$

It follows that

$$P(Q(t) = j, \sigma(t) = 1 | Q(0) = m, \sigma(0) = 1) = \tag{7}$$
$$= \quad e^{-(\lambda+\mu)t} \rho^{\frac{j-m}{2}} \left[I_{|j-m|}(2t\sqrt{\lambda\mu}) - I_{j+m}(2t\sqrt{\lambda\mu}) \right]$$
$$+ \quad \sum_{n\geq 0} \sum_{a\geq 0} \sum_{i\geq 1} \left(\frac{\pi}{\mu}\right)^{\nu n} \left(1 - \frac{\pi}{\mu}\right)^{a} \rho^{\frac{j-i-a-\nu n-m}{2}} \binom{a+\nu n-1}{a} \times$$
$$\times \quad \int_0^t h_i(t-s) e^{-(\lambda+\mu)s} \left[I_{a+\nu n+m+|j-i|}(2s\sqrt{\lambda\mu}) - I_{a+\nu n+m+j+i}(2s\sqrt{\lambda\mu}) \right] ds.$$

Similarly we find for the case that the server is on a vacation at time t:

$$P(Q(t) = j, \sigma(t) = 0 | Q(0) = m, \sigma(0) = 1) = \tag{8}$$
$$= \quad f^{(m)}(t) * H_j(t) * \sum_{n\geq 0} g^{(n)}(t)$$
$$= \quad \sum_{n\geq 0} \sum_{a\geq 0} \binom{a+\nu n-1}{a} \left(\frac{\pi}{\mu}\right)^{\nu n} \left(1 - \frac{\pi}{\mu}\right)^{a} (m+a+\nu n) \times$$
$$\times \quad \rho^{-\frac{m+a+\nu n}{2}} \int_0^t H_j(t-s) e^{-(\lambda+\mu)s} I_{a+m+\nu n}(2s\sqrt{\lambda\mu}) \frac{ds}{s},$$

where $H_j(t) = \int_t^\infty h_j(s)\, ds$.

4 References

[1] Takagi H. (1990) Time-dependent analysis of M/G/1 vacation models with exhaustive service. *Queueing Systems*, **6**, 4, 369-390.