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# A Note on M/M/1 Queues with Gamma-Distributed Vacations



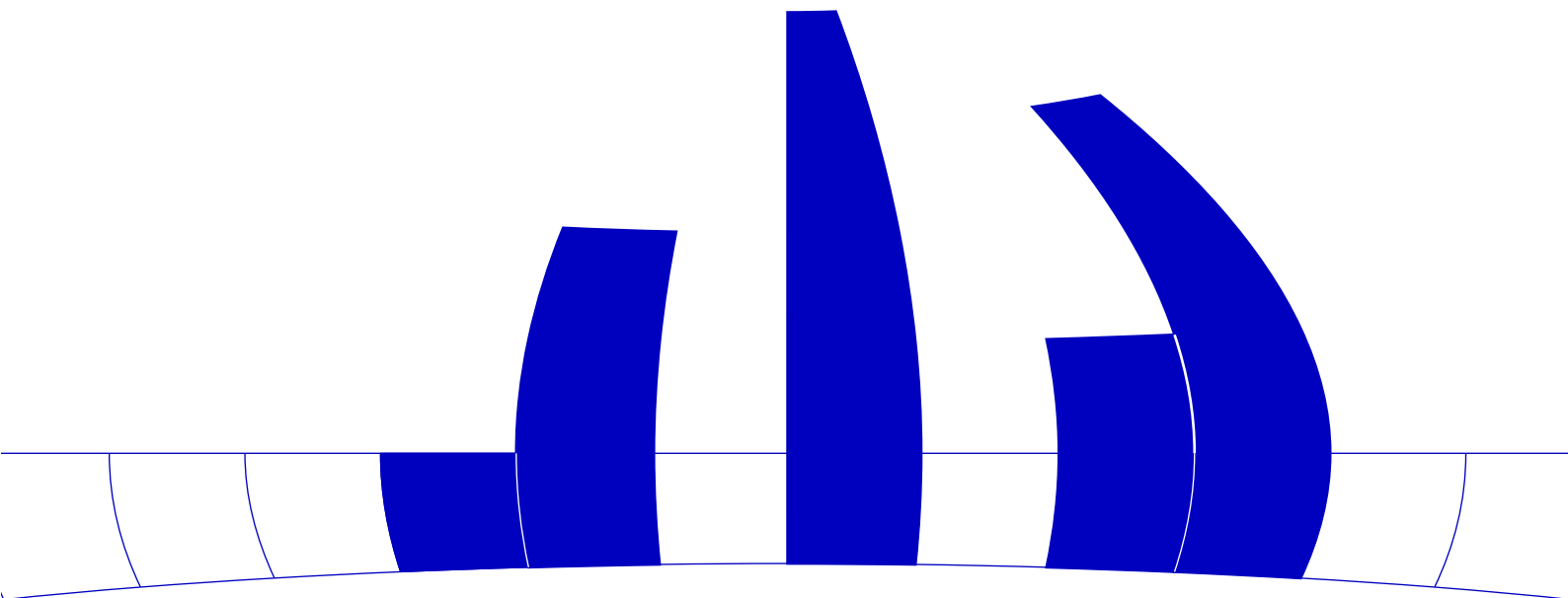
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# A note on M/M/1 queues with gamma-distributed vacations

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## Abstract

In this note we derive a simple formula for the density of  $S_n = \sum_{i=1}^n T_i$ , where  $T_i$  is the duration of the  $i$ -th busy cycle in an M/M/1 queueing model with gamma distributed server vacations. Using this result we find the distribution of the queue length by standard renewal arguments.

## 1 Introduction

Recently Takagi (1990) has presented a paper in which the Laplace transforms of various queueing characteristics for M/G/1 systems with server vacations are given. However, it is by no means trivial to determine the inverse transforms, even in the special case of M/M/1. In this note we discuss the problem of determining the transient distribution of the queue length in an M/M/1 queueing model where customers arrive with rate  $\lambda$ , are served with rate  $\mu$  and the server goes on a vacation whenever the queue becomes empty. We assume that the vacations are mutually independent random variates with common density

$$w(t) = \frac{e^{-\pi t} \pi^\nu t^{\nu-1}}{(\nu-1)!},$$

where  $\pi$  and  $\nu$  are arbitrary reals, satisfying  $\pi > 0$  and  $\nu \geq 1$ . The key result is the density function of  $S_n = \sum_{i=1}^n T_i$ , where  $T_i$  is the duration of the  $i$ -th busy cycle. Rather surprisingly it will turn out, that this density of particularly simple form.

## 2 The busy cycle

We define a busy cycle as a vacation followed by a completed busy period. It is assumed that vacations can start only at those time points where busy periods terminate, thus at the beginning of a vacation there are no customers in the queue. A cycle may be degenerate, which will happen, if there are no arrivals during a vacation. In such a case we assume that the busy period following this vacation has length zero.

Let  $T_i$  denote the length of a cycle. Since the lengths of the server vacations are i.i.d. random variates and independent of the service process, it follows, that also the  $T_i$  are mutually independent with density function  $g(t)$ . Some reflection shows that under these conditions the Laplace transform (L.T.) of  $g(t)$ , viz.  $g^*(s)$  is given by

$$g^*(s) = \sum_{a \geq 0} h_a^*(s) \xi^a(s), \quad (1)$$

where  $h_a^*(s)$  is the L.T. of

$$h_a(t) = w(t) \frac{e^{-\lambda t} (\lambda t)^a}{a!},$$

and  $\xi(s)$  is the L.T. of the density of a busy period initiated by a single customer. It is well known (see e.g. Prabhu (1965, p. 8)), that

$$\xi(s) = \frac{1}{2\lambda} [s + \lambda + \mu - \sqrt{(s + \lambda + \mu)^2 - 4\lambda\mu}], \quad (2)$$

where the square root has to be taken so that its real part is positive. Besides other functional relations the function  $\xi(s)$  satisfies

$$\xi(s)\eta(s) = \frac{\mu}{\lambda}, \quad (3)$$

where

$$\eta(s) = \frac{1}{2\lambda} [s + \lambda + \mu + \sqrt{(s + \lambda + \mu)^2 - 4\lambda\mu}]. \quad (4)$$

Equation (1) can be simplified further to yield

$$g^*(s) = w^*(s + \lambda - \lambda\xi(s)), \quad (5)$$

a result, which may be found in Takagi (1990).

Henceforth we assume that the server vacations have a gamma density

$$w(t) = \frac{e^{-\pi t} \pi^\nu t^{\nu-1}}{(\nu-1)!},$$

where  $\pi$  is real positive and  $\nu$  is any real number  $\geq 1$ .

Let  $S_n = \sum_{i=0}^n T_i$ . The density of  $S_n$  is given by the  $n$ -fold convolution of  $g(t)$ , which we denote by  $g^{(n)}(t)$ . We will show that this density has the following interesting form:

$$g^{(n)}(t) = \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a \geq 0} \binom{a + \nu n - 1}{a} f^{(a+\nu n)}(t) \left(1 - \frac{\pi}{\mu}\right)^a, \quad (6)$$

with

$$f^{(a+\nu n)}(t) = e^{-(\lambda+\mu)t} \rho^{-\frac{a+\nu n}{2}} \frac{a + \nu n}{t} I_{a+\nu n}(2t\sqrt{\lambda\mu}),$$

and  $I_a(2t\sqrt{\lambda\mu})$  denotes the modified Bessel function of (possibly) fractional order  $a$ .

To prove (6) we note that according to (5) the L.T. of  $g^{(n)}(t)$  is given by

$$g^{n*}(s) = \left(\frac{\pi}{\pi + \lambda + s - \lambda\xi(s)}\right)^{\nu n}.$$

Because of (3), this can be rewritten as

$$\begin{aligned} g^{n*}(s) &= \left(\frac{\pi}{\pi - \mu + \mu/\xi(s)}\right)^{\nu n} \\ &= \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a \geq 0} \binom{a + \nu n - 1}{a} \xi^{a+\nu n}(s) \left(1 - \frac{\pi}{\mu}\right)^a. \end{aligned}$$

The result follows upon inverting  $\xi^{a+\nu n}(s)$ .

Some remarks are in order concerning (6). First,  $g^{(n)}(t)$  turns out to be a weighted sum of densities  $f^{(a+\nu n)}(t)$ . These densities are proper, if  $\lambda < \mu$  and equal the densities of busy periods initiated by  $a + \nu n$  customers, if  $a + \nu n$  is a positive integer. However, such an interpretation is in general not possible, if  $\nu$  is an arbitrary real number  $\geq 1$ . Furthermore, we note, that when  $\pi < \mu$ , the weights in (6) are the probabilities of a negative binomial distribution. And finally, (6) remains valid even in the special case  $\pi = \mu$ , where (6) simplifies to

$$g^{(n)}(t) = e^{-(\lambda+\mu)t} \rho^{-\frac{\nu n}{2}} \frac{\nu n}{t} I_{\nu n}(2t\sqrt{\lambda\mu}).$$

### 3 The distribution of the queue length

Formula (6) allows us to state the distribution of the queue length in comparatively simple terms. Let  $Q(t)$  denote the number of customers in the queue at time  $t$  and define an indicator function  $\sigma(t)$  by

$$\sigma(t) = \begin{cases} 1 & \text{if the server is busy at time } t \\ 0 & \text{otherwise} \end{cases}$$

Furthermore let  $r_j(t)$  denote the probability that at time  $t$  the server is busy for the first time with  $j$  customers waiting, provided at time zero a vacation started. Then we have

$$r_j(t) = \sum_{i \geq 1} h_i(t) * {}^0P_{ij}(t),$$

where  $*$  denotes convolution and

$${}^0P_{ij}(t) = P(Q(t) = j, Q(s) > 0, 0 < s < t | Q(0) = j).$$

The functions  ${}^0P_{ij}(t)$  are well known and given for instance in Prabhu (1965, p. 15):

$${}^0P_{ij}(t) = e^{-(\lambda+\mu)t} \rho^{\frac{i-j}{2}} \left[ I_{|j-i|}(2t\sqrt{\lambda\mu}) - I_{i+j}(2t\sqrt{\lambda\mu}) \right]$$

Let us now assume that the server is busy at time zero with  $m$  customers waiting. Then the event  $\{Q(t) = j, \sigma(t) = 1\}$  implies that the server had  $n \geq 0$  vacations in  $(0, t)$ , where the last vacation terminated at some time  $0 < s < t$  with a positive number of customers having arrived during that vacation. Furthermore the server is continuously busy in  $(s, t)$ . Hence we have

$$\begin{aligned} P(Q(t) = j, \sigma(t) = 1 | Q(0) = m, \sigma(0) = 1) &= \\ &= {}^0P_{mj}(t) + f^{(m)}(t) * r_j(t) * \sum_{n \geq 0} g^{(n)}(t). \end{aligned}$$

This formula may be simplified further by observing that

$$f^{(m)}(t) * g^{(n)}(t) = \left(\frac{\pi}{\mu}\right)^{\nu n} \sum_{a \geq 0} \binom{a + \nu n - 1}{a} f^{(a+m+\nu n)}(t) \left(1 - \frac{\pi}{\mu}\right)^a,$$

and

$$\begin{aligned} f^{(a+m+\nu n)}(t) * {}^0P_{ij}(t) &= \\ &= e^{-(\lambda+\mu)t} \rho^{\frac{j-i-a-\nu n-m}{2}} \left[ I_{a+\nu n+m+|j-i|}(2t\sqrt{\lambda\mu}) - I_{a+\nu n+m+j+i}(2t\sqrt{\lambda\mu}) \right]. \end{aligned}$$

It follows that

$$\begin{aligned} P(Q(t) = j, \sigma(t) = 1 | Q(0) = m, \sigma(0) = 1) &= \tag{7} \\ &= e^{-(\lambda+\mu)t} \rho^{\frac{j-m}{2}} \left[ I_{|j-m|}(2t\sqrt{\lambda\mu}) - I_{j+m}(2t\sqrt{\lambda\mu}) \right] \\ &+ \sum_{n \geq 0} \sum_{a \geq 0} \sum_{i \geq 1} \left( \frac{\pi}{\mu} \right)^{\nu n} \left( 1 - \frac{\pi}{\mu} \right)^a \rho^{\frac{j-i-a-\nu n-m}{2}} \binom{a+\nu n-1}{a} \times \\ &\times \int_0^t h_i(t-s) e^{-(\lambda+\mu)s} \left[ I_{a+\nu n+m+|j-i|}(2s\sqrt{\lambda\mu}) - I_{a+\nu n+m+j+i}(2s\sqrt{\lambda\mu}) \right] ds. \end{aligned}$$

Similarly we find for the case that the server is on a vacation at time  $t$ :

$$\begin{aligned} P(Q(t) = j, \sigma(t) = 0 | Q(0) = m, \sigma(0) = 1) &= \tag{8} \\ &= f^{(m)}(t) * H_j(t) * \sum_{n \geq 0} g^{(n)}(t) \\ &= \sum_{n \geq 0} \sum_{a \geq 0} \binom{a+\nu n-1}{a} \left( \frac{\pi}{\mu} \right)^{\nu n} \left( 1 - \frac{\pi}{\mu} \right)^a (m+a+\nu n) \times \\ &\times \rho^{-\frac{m+a+\nu n}{2}} \int_0^t H_j(t-s) e^{-(\lambda+\mu)s} I_{a+m+\nu n}(2s\sqrt{\lambda\mu}) \frac{ds}{s}, \end{aligned}$$

where  $H_j(t) = \int_t^\infty h_j(s) ds$ .

## 4 References

- [1] Takagi H. (1990) Time-dependent analysis of M/G/1 vacation models with exhaustive service. *Queueing Systems*, **6**, 4, 369-390.