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On random walks with barriers and their application to queues

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Abstract

The n-step transition probabilities of a random walk with two barriers, each being either reflecting or absorbing are considered on the basis of a simple renewal argument. The relation of these walks to queueing problems is pointed out and the distributions of the queue length in the finite capacity case, the same during a busy period and of the maximum queue length are derived for discrete time models. By taking the limit the solutions of continuous time models are derived, verifying some known results.

1 Introduction

In this paper we will be concerned with the one-dimensional random walk $Z(n)$ having state space the set of all integers and one-step transition probabilities

$$\begin{aligned} P(Z(n)=k+1|Z(n-1)=k) &= \alpha \\ P(Z(n)=k-1|Z(n-1)=k) &= \gamma \\ P(Z(n)=k|Z(n-1)=k) &= \beta, \end{aligned}$$

where $\beta = 1-\alpha-\gamma$. It is well known that there is an intimate connection between the random walk $Z(n)$ and the discrete time queueing process $Q(n)$, in particular the following relation holds:

$$Q(n) = Z(n) - \min_{0\le i\le n} Z(i) \wedge 0,$$

which states that $Q(n)$ is derived from $Z(n)$ by introducing a reflecting barrier at zero.

Suppose now that $Z(n)$ is restricted by two barriers, one at zero and the other located at $h > 0$, where each one may be either reflecting or absorbing. Here four possible cases arise and each has its own interpretation in a queueing context. In the simplest case there are two absorbing barriers, one at zero and the other one located at h. The transitions of $Z(n)$ between these barriers coincide with the transitions of the process $Q(n)$ during a period where the service facility is continuously busy and the maximum queue length is less than h. As the second case we will consider a reflecting barrier at zero and an absorbing barrier at h. This time $Z(n)$ is equivalent to a queueing process with maximum queue length less than h and such that the server need not be continuously busy. In this context we will also discuss the distribution of the stopping time τ_{mh}, the time the process $Q(n)$ requires to reach a maximum value of h for the first time, provided that there are m customers initially waiting. In the third case there will be an absorbing barrier at zero and a reflecting barrier at h. Now $Z(n)$ corresponds to a queueing process $Q(n)$, which is continuously busy up to time n and has a waiting room with finite capacity h. In the remaining fourth case there will be two reflecting barriers, located at zero and at h. This arrangement gives rise to the distribution of the queue length of a queueing process with finite waiting room capacity h, the server need not be continuously busy.

At a first glance it seems that the introduction of various barriers is a purely formal procedure, changing only the boundary conditions of the system of partial difference equations, which has to be solved in order to find the distribution of $Q(n)$. However, the presence of barriers may be given an interesting probabilistic interpretation, which provides us with additional insight into the structure of the process $Q(n)$ and with a simple method of solving all four possible cases mentioned above.

To give this interpretation and to outline the main argument which we will use in subsequent sections, the case where $Z(n)$ is restricted by one reflecting barrier at zero, thus giving rise the the simple queueing process $Q(n)$, will be discussed shortly.

Suppose that $Z(0) = m > 0$, then it is apparent that the transitions of $Z(n)$ and $Q(n)$ coincide as long as $Z(n)$ remains positive. At the time when $Z(n)$ touches the barrier for the first time, a busy period of $Q(n)$ terminates and therefore the stopping time

$$T_{m0} = \inf\{n : Z(n) = 0 | Z(0) = m\}$$

equals the duration of a busy period initiated by m customers. The end of a busy period marks the beginning of an idle period which will terminate with the arrival of the next customer. Let A_1 denote the length of this idle period. It equals the number of trials up to the first success in a sequence of Bernoulli experiments having success probability α and hence has a geometric distribution. With the arrival of the next customer the random walk $Z(n)$ starts anew and continues up to the time when the server becomes free again. Thus the queueing process $Q(n)$ exhibits a typical repetitive pattern. It is regenerative with renewal points those time instants, where the queue becomes empty. Any two consecutive renewal points include exactly one idle period.

Let us now assume that there are $i \geq 0$ completed idle periods and consider the event $\{Q(n) = k|Q(0) = m\}$. This event has probability

$$P(Z(n) = k, T_{m0} > n|Z(0) = m) \qquad \text{if } i = 0,$$

and

$$\begin{aligned} &P(T_{m0} = n) * [P(A_1 = n) * P(T_{10} = n)]^{i-1\,*} * \\ &* \;\; P(A_1 = n) * P(Z(n) = k, T_{10} > n|Z(0) = 1) \qquad \text{if } i > 0, \end{aligned}$$

where "$*$" denotes convolution and $(\;)^{i*}$ means i-fold convolution. Noting that by the Markov property of $Z(n)$ we have

$$P(T_{a0} = n) * P(T_{b0} = n) = P(T_{a+b,0} = n)$$

and

$$P(A_1 = n)^{i*} = P(A_i = n),$$

where A_i is the number of trials up to the i-th success in a sequence of Bernoulli experiments, we obtain for $k > 0$ after summing on i:

$$\begin{aligned} &P(Q(n) = k|Q(0) = m) = \\ &= \;\; P(Z(n) = k, T_{m0} > n|Z(0) = m) + \\ &\quad\; + P(Z(n) = k, T_{10} > n|Z(0) = 1) * \sum_{i \geq 0} P(A_{i+1} = n) * P(T_{m+i,0} = n). \end{aligned}$$

Since A_i has a negative binomial distribution and the distribution of T_{m0} is given by

$$P(T_{m0} = n) = \gamma P(Z(n-1) = 1, T_{m0} > n-1|Z(0) = m),$$

the transient distribution of $Q(n)$ is primarily determined by the zero-avoiding transition probabilities

$$P(Z(n) = k, T_{m0} > n | Z(0) = m). \quad (1)$$

The key point in the above analysis is that it takes into account the structure of the process by splitting the sample paths into repetitive patterns at renewal points and possibly identifying a distribution on which the determination of the transient distribution depends.

In this paper we will follow the same approach for two-barrier cases. Clearly the zero-avoiding transition probabilities will be replaced by $\{0, h\}$-avoiding transition probabilities, which are needed in each of the four situations.

In Karlin and McGregor (1959) this type of random walks has been treated through the method of spectral decomposition. This method may be used to determine the transient solution of general birth-death processes. It exhibits its full power if the transition probability matrices are of infinite dimension which gives rise to continuous spectra. In the finite dimensional case it reduces to the determination of eigenvalues and eigenvectors of a tridiagonal matrix, which is not always the simplest way to find a solution. This method works from the top to the bottom in the sense that first the one-step transition matrix is set up, then its eigenvalues, eigenvectors and their norms are determined leading to the final evaluation of transition probabilities. The results so obtained for one type of barriers cannot be utilized for another type for which the whole procedure has to be repeated right from the beginning. In this sense it is unable to exploit the structural properties of the processes.

In contrast our method proceeds from the bottom to the top, as explained hereafter. First we look for appropriate renewal points, and then we decompose the sample paths according to those points into subsegments whose generating functions are known from the two-absorbing-barrier case. Our method does not only give the solutions to the various cases, but also utilizes the structure common to two-barrier situations. Furthermore it is simple and unifying in the sense that only one generating function is needed, from which all results are derived in a straight forward manner. Its use is not restricted to the discussion of simple random walks, but also applies to more general Markov processes. It may further be noted that our technique resembles a powerful principle from enumerative combinatorics: in order to count a set of configurations one usually looks for a decomposition into simpler subconfigurations. This gives rise to a factorization of the generating function carrying the enumerative information into simpler functions, which may be known (Goulden and Jackson (1983), pp. 34).

In Mohanty and Panny (1990a) an elegant geometric-combinatoric method is suggested for the random walk with one reflecting barrier at zero. It is unfortunate that the same method cannot be adapted for two-barrier cases, if at least one is reflecting. In another paper (Mohanty and Panny (1990b)) for the same one-barrier problem, they started with the generating function of the random walk having a reflecting barrier at zero and an absorbing barrier at h. However, they neither used the structure of the process nor gave the solution to the two-barrier problem.

The paper is organized as follows: in the next section the basic generating function is given and used to derive the transient solutions of the discrete time queueing processes described above. In the last section we will show how these results translate to continuous time random walks.

Usually solutions for continuous time models are considered and are used as approximations for discrete time models. What we have achieved in this paper is to provide exact transient solutions for discrete time models, and by taking the limit, the solutions of continuous time models are derived, verifying some known results.

2 The results in discrete time

21 The basic generating function

Let $p_n(h,m,k) = P(Z(n)=k, 0<Z(i)<h, i=0,\ldots,n|Z(0)=m)$ and define the pgf.

$$G_{mk}(s) = \sum_{n\geq 0} s^n p_n(h,m,k).$$

Then it can be shown that (Panny (1984))

$$\begin{aligned} G_{mk}(s) &= \frac{(\alpha v^2+\beta v+\gamma)(\rho v)^{k-m}(1-(\rho v^2)^{h-k})(1-(\rho v^2)^m)}{\gamma(1-\rho v^2)(1-(\rho v^2)^h)} \qquad (k\geq m)\ (2) \\ &= \frac{(\alpha v^2+\beta v+\gamma)v^{m-k}(1-(\rho v^2)^{h-m})(1-(\rho v^2)^k)}{\gamma(1-\rho v^2)(1-(\rho v^2)^h)} \qquad (k\leq m), \end{aligned}$$

where the substitution $s = (\alpha v + \beta + \gamma/v)^{-1}$ has been used and $\rho = \alpha/\gamma$. Two useful pgf.s may be derived readily from (2): define the stopping times

$$\begin{aligned} T_{m0} &= \inf\{n : Z(n) = 0 | Z(0) = m, Z(i) < h, i = 0, \ldots, n\} \\ T_{mh} &= \inf\{n : Z(n) = h | Z(0) = m, Z(i) > 0, i = 0, \ldots, n\}, \end{aligned}$$

with $H_{m0}(s) = \sum_{n \geq 0} s^n P(T_{m0} = n)$ and $H_{mh}(s) = \sum_{n \geq 0} s^n P(T_{mh} = n)$. Then we have

$$\begin{aligned} H_{m0}(s) &= s\gamma G_{m1}(s) \\ &= \frac{v^m(1 - (\rho v^2)^{h-m})}{1 - (\rho v^2)^h}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} H_{mh}(s) &= s\gamma G_{m,h-1}(s) \\ &= \frac{(\rho v)^{h-m}(1 - (\rho v^2)^m)}{1 - (\rho v^2)^h}. \end{aligned} \tag{4}$$

For brevity we omit the arguments in the pgf.s, for example we write G_{mk} for $G_{mk}(s)$.

22 Absorbing barriers at 0 and h

This case has been dealt with several times in the literature, e.g. by Kemperman (1961) for random walks with only two types of steps. We note that the following identity holds:

$$\begin{aligned} &\{Z(n) = k, 0 < Z(i) < h, 0 \leq i \leq n | Z(0) = m\} \equiv \\ &\quad \equiv \{Q(n) = k, 0 < Q(i) < h, 0 \leq i \leq n | Q(0) = m\} \\ &\quad \equiv \{Q(n) = k, \max_{0 \leq i \leq n} Q(i) < h, Q(i) > 0, 0 \leq i \leq n | Q(0) = m\}. \end{aligned}$$

Thus

$$P(Q(n) = k, \max_{0 \leq i \leq n} Q(i) < h, Q(i) > 0, 0 \leq i \leq n | Q(0) = m) = p_n(h, m, k).$$

Expanding (2) into partial fractions and using Theorem 1 (pp. 9-10) in Panny (1984), an explicit expression for $p_n(h, m, k)$ may be given as follows:

$$p_n(h, m, k) = \frac{2}{h} \rho^{\frac{k-m}{2}} \sum_{\nu=1}^{h-1} \sin \frac{(h-m)\nu\pi}{h} \sin \frac{(h-k)\nu\pi}{h} (\beta + 2\sqrt{\alpha\gamma} \cos \frac{\nu\pi}{h})^n. \tag{5}$$

It will be both, convenient and instructive, to have an alternative representation of $p_n(h,m,k)$ in terms of generalized trinomial coefficients, which are defined by

$$\binom{n;\alpha,\beta,\gamma}{k} = [v^k](\alpha v^2+\beta v+\gamma)^n \tag{6}$$

and are, as we shall see later, the direct discrete time analogues of the modified Bessel functions. So in terms of generalized trinomial coefficients we have

$$p_n(h,m,k) = \sum_{\nu=-\infty}^{\infty} \rho^{h\nu}\left[\binom{n;\alpha,\beta,\gamma}{n+k-m-2\nu h} - \rho^{h-m}\binom{n;\alpha,\beta,\gamma}{n+k+m-2h(\nu+1)}\right], \tag{7}$$

which may be derived from (2) by applying Cauchy's integral theorem. This formula has a striking combinatorial flavour, since it can be derived by repeatedly applying the reflection principle to the sample paths of $Z(n)$. However, expression (7) cannot be derived directly by the method of spectral decomposition.

23 Absorbing barrier at h and reflecting barrier at 0

As we remarked in the introduction, in this case the n-step transition probabilities of the random walk $Z(n)$ equal

$$P(Q(n)=k, \max_{0\le i\le n} Q(i) < h | Q(0)=m). \tag{8}$$

Let us exploit the renewal properties of the process $Q(n)$. We note that a convenient set of renewal points are those time instants where the queue becomes empty. From the discussion in the introduction it is clear that the lengths of idle periods have a geometric distribution with success probability α. Let $W(s)$ be the corresponding pgf. Then upon using the substitution $s=(\alpha v+\beta+\gamma/v)^{-1}$ we have

$$W(s) = \frac{\alpha s}{1-(1-\alpha)s} = \frac{\rho v}{\rho v^2 - v + 1}.$$

Suppose now m and k are positive and let $S(h,m,k)$ be the pgf. of (8). Then

$$S(h,m,k) = \sum_{i\ge 0} S_i(h,m,k),$$

where $S_i(h,m,k)$ is the pgf. of (8), given that there are i completed idle periods. By using the renewal properties, we get

$$S_i(h,m,k) = H_{m0} W^i H_{10}^{i-1} G_{1k} \qquad (i \geq 1) \tag{9}$$

and

$$S_0(h,m,k) = G_{mk}. \tag{10}$$

Hence we find

$$\begin{aligned} S(h,m,k) &= G_{mk} + H_{m0} G_{1k} W \sum_{i \geq 0} W^i H_{10}^i \\ &= G_{mk} + \frac{H_{m0} G_{1k} W}{1 - H_{10} W}. \end{aligned} \tag{11}$$

Similar reasoning shows that

$$\begin{aligned} S(h,m,0) &= \frac{H_{m0} W}{\alpha s(1 - H_{10} W)} \\ S(h,0,k) &= \frac{G_{1k} W}{1 - H_{10} W} \\ S(h,0,0) &= \frac{W}{\alpha s(1 - H_{10} W)} \end{aligned} \tag{12}$$

The generating functions occurring in the formulas above are all known (see (2), (3)). Inserting their expressions we obtain for $m \leq k$:

$$S(h,m,k) = \frac{(\rho v)^{k-m}(\alpha v^2 + \beta v + \gamma)[1 - (\rho v^2)^{h-k}][1 - v + v(1 - \rho v)(\rho v^2)^m]}{\gamma(1 - \rho v^2)[1 - v + v(1 - \rho v)(\rho v^2)^h]}, \tag{13}$$

and for $m \geq k$:

$$S(h,m,k) = \frac{v^{m-k}(\alpha v^2 + \beta v + \gamma)[1 - (\rho v^2)^{h-m}][1 - v + v(1 - \rho v)(\rho v^2)^k]}{\gamma(1 - \rho v^2)[1 - v + v(1 - \rho v)(\rho v^2)^h]}. \tag{14}$$

It is easily checked that (13) and (14) cover also the cases where m and (or) k are equal to zero. It is possible to expand (13) and (14) into partial fractions, however, this requires the knowledge of the roots of the polynomial equation

$$1 - v + \rho^h v^{2h+1} - \rho^{h+1} v^{2h+2} = 0.$$

Unfortunately the roots (and hence the eigenvalues of the transition matrix), except for the trivial case $\rho = 1$, cannot be given in closed form. However, for completeness, we will sketch how a partial fraction expansion may be obtained. Using the substitution $v = \rho^{-1/2}e^{\theta i}$, we find for $m \geq k$:

$$S(h,m,k) = \frac{\rho^{\frac{m-k-}{2}}(\beta + 2\sqrt{\alpha\gamma}\cos\theta)\sin(h-k)\theta\,[\sin(m+1)\theta - \rho^{-1/2}\sin m\theta]}{\gamma\sin\theta\,[\sin(h+1)\theta - \rho^{-1/2}\sin h\theta]}. \tag{15}$$

The roots θ_ν of the denominator have to be determined numerically (one of the angles θ_ν will be complex if $\rho^{1/2} < h/(h+1)$) and after expansion into partial fractions:

$$P(Q(n) = k, \max_{0\leq i\leq n} Q(i) < h | Q(0) = m) = \tag{16}$$

$$= \quad -2\rho^{\frac{k-m}{2}} \sum_{\nu=1}^{h} \frac{(\beta + 2\sqrt{\alpha\gamma}\cos\theta_\nu)^n \sin(h-k)\theta_\nu\,[\sqrt{\alpha}\sin(m+1)\theta_\nu - \sqrt{\gamma}\sin m\theta_\nu]}{\sqrt{\alpha}(h+1)\cos(h+1)\theta_\nu - \sqrt{\gamma}h\cos h\theta_\nu},$$

which holds also for $m \leq k$.

Root finding may be avoided if we expand the denominator of (13) directly. For this purpose we note that the function $(1 - v + v(1-\rho v)(\rho v^2)^h)^{-1}$ is analytic in a disc around the origin. Therefore it has an expansion of the form

$$(1 - v + v(1-\rho v)(\rho v^2)^h)^{-1} = \sum_{a\geq 0} d_{a,h} v^a \qquad (|v| \leq 1),$$

where an application of the binomial theorem yields

$$d_{a,h} = (-\rho)^a \sum_{i\leq a}\sum_{j\geq 0} \binom{i}{j}\binom{j}{a-i-2hj}(-1)^{i+j}\rho^{-i-hj}. \tag{17}$$

Let us now expand (14) by means of Cauchy's integral theorem. We obtain:

$$P(Q(n) = k, \max_{0\leq i\leq n} Q(i) < h | Q(0) = m) = \tag{18}$$

$$= \quad \frac{1}{2\pi i}\oint \frac{v^{m-k}}{v^{n+1}} \frac{(\alpha v^2 + \beta v + \gamma)^n [1 - (\rho v^2)^{h-m}][1 - v + v(1-\rho v)(\rho v^2)^k]}{1 - v + v(1-\rho v)(\rho v^2)^h}\,dv$$

$$= \quad \sum_{a\geq 0} d_{a,h}\left[\binom{n;\alpha,\beta,\gamma}{n-m+k-a} - \binom{n;\alpha,\beta,\gamma}{n-m+k-a-1}\right]$$

$$-\rho^{h-m}\sum_{a\geq 0} d_{a,h}\left[\binom{n;\alpha,\beta,\gamma}{n+m+k-2h-a} - \binom{n;\alpha,\beta,\gamma}{n+m+k-2h-a-1}\right]$$

$$+\rho^{k}\sum_{a\geq 0} d_{a,h}\left[\binom{n;\alpha,\beta,\gamma}{n-m-k-a-1} - \rho\binom{n;\alpha,\beta,\gamma}{n-m-k-a-2}\right]$$

$$-\rho^{h-m+k}\sum_{a\geq 0} d_{a,h}\left[\binom{n;\alpha,\beta,\gamma}{n+m-k-2h-a-1} - \rho\binom{n;\alpha,\beta,\gamma}{n+m-k-2h-a-2}\right].$$

It can be shown that (18) covers also the case $m \leq k$.

At this point it is worthwhile to remark that in the spectral analysis method the derivation of the pgf. is not explicit and therefore the method may not lead to (18) directly. For the same reason the method cannot utilize the information obtained on various segments in the pgf. and thus is forced to start all over again when a new situation arises, as illustrated in section 2.5. However, our approach, in which the explicit derivation of the pgf. is important, uses the basic pgf. (2) and its special cases (3) and (4) in deriving (13), which in turn will appear in section 2.5.

From the pgf. $S(h,m,k)$ additional interesting information may be obtained. Let $\tau_{mh} = \inf\{n : Q(n) = h | Q(0) = m\}$, the time until a maximum queue length of h is reached for the first time. We note that the pgf. of τ_{mh} equals $s\alpha S(h,m,h-1)$ and therefore $P(\tau_{mh} < \infty) = 1$, since $S(h,m,h-1)|_{s=1} = 1$. The probability function of τ_{mh} is clearly

$$P(\tau_{mh} = n) = \alpha P(Q(n-1) = h-1, \max_{0\leq i\leq n-1} Q(i) < h | Q(0) = m),$$

and therefore an explicit expression may be obtained from (18). For the expected value of τ_{mh} we find

$$\begin{aligned} E(\tau_{mh}) &= \frac{d}{ds} s\alpha S(h,m,h-1)|_{s=1} \\ &= \frac{d}{ds}\frac{(\rho v)^{h-m}(1-v+v(1-\rho v)(\rho v^2)^m)}{1-v+v(1-\rho v)(\rho v^2)^h}|_{s=1}. \end{aligned}$$

Now

$$\frac{dv}{ds} = \frac{(\alpha v^2+\beta v+\gamma)^2}{\gamma(1-\rho v^2)},$$

and therefore

$$E(\tau_{mh}) = \frac{\rho^m - \rho^h - \rho^{h+m}(h-m)(1-\rho)}{\gamma(1-\rho)^2\rho^{h+m}} \qquad (\rho \neq 1). \tag{19}$$

The results of this section have, as far as we know, not been reported in the literature until now.

24 Absorbing barrier at 0 and reflecting barrier at h

In the case where the random walk $Z(n)$ is restricted by a reflecting barrier at h and an absorbing barrier at 0, it behaves like a queueing process with finite capacity h during a period where the server is continuously busy. To fix notation we write $Q_h(n)$ for the queueing process with finite capacity h at time n. Then

$$P(Q_h(n) = k, Q(i) > 0, 0 \leq i \leq n | Q_h(0) = m)$$

can be simply derived from the results of the previous section by observing the following duality relation: consider the reversed sample paths of the queueing process $Q(n)$ restricted by an absorbing barrier at h and a reflecting barrier at 0. Then it is immediately seen that

$$\begin{aligned} P(Q(n) = k, \max_{0 \leq i \leq n} Q(i) | Q(0) = m) &= \\ = \; P(Q_h^\star(n) = h - m, Q_h^\star(i) > 0, 0 \leq i \leq n | Q_h^\star(0) = h - k), \end{aligned} \tag{20}$$

where the process $Q_h^\star(n)$ is obtained from $Q_h(n)$ by interchanging the arrival and departure probabilities α and γ.

Let us denote the length of a busy period under finite capacity h by τ_{m0}^h. Its distribution is found by arguments similar to those we used to derive the distribution of τ_{mh} in the previous section and by exploiting the duality relation (20). In particular we have

$$\{\tau_{m0}^h = n\} \equiv \{\tau_{h-m,h}^\star = n\},$$

where the star indicates that the probabilities α and γ have to be interchanged. Therefore

$$\begin{aligned} E(\tau_{m0}^h) &= E(\tau_{h-m,h}^\star) \\ &= \frac{\rho^{h+m+1} - \rho^{h+1} + m\rho^m(1-\rho)}{\gamma(1-\rho)^2\rho^m} \qquad (\rho \neq 1). \end{aligned} \tag{21}$$

25 Reflecting barriers at 0 and h

In this case the renewal argument may be formulated in terms of the random walk $Z(n)$ restricted by a reflecting barrier at zero, as it was discussed in section 2.3. As renewal points we choose those time instants where $Z(n)$ touches the barrier at h. At the moment when $Z(n)$ touches this barrier, the waiting room is full and the system closes in the sense that customers arriving now are not allowed to enter the system and disappear. The system remains closed until the next departure which frees space in the waiting room. It is clear that the time the system remains closed has a geometric distribution with pgf.

$$V(s) = \frac{\gamma s}{1-(1-\gamma)s} = \frac{v}{\rho v^2 - \rho v + 1}, \tag{22}$$

where again $s = (\alpha v + \beta + \gamma/v)^{-1}$. From section 2.3 it is known that the pgf. of τ_{mh}, the time required to reach a maximum value of h for the first time, is $\alpha s S(h,m,h-1)$. Suppose now, that the system has been closed $i \geq 0$ times and the sample path of the queueing process $Q_h(n)$ leads from m to k, where $m, k < h$. Then the corresponding pgf. is given by

$$\begin{aligned} T_i(h,m,k) &= S(h,m,h-1)(\alpha s V)^i S^{i-1}(h,h-1,h-1)S(h,h-1,k) \quad (i \geq 1) \\ T_0(h,m,k) &= S(h,m,k). \end{aligned} \tag{23}$$

Let $T(h,m,k) = \sum_{i \geq 0} T_i(h,m,k)$. It follows that $T(h,m,k)$ is the pgf. of $P(Q_h(n) = k | Q_h(0) = m)$, in particular

$$T(h,m,k) = S(h,m,k) + \frac{s\alpha S(h,m,h-1)\, V\, S(h,h-1,k)}{1 - s\alpha\, V\, S(h,h-1,h-1)}. \tag{24}$$

This expression may be simplified further by observing that

$$(\rho v^2)^a R(h-a) = R(h) - (1-v)(1-(\rho v^2)^a)$$

and

$$(\rho v^2 - \rho v + 1)R(a) = (1-v)(1-\rho v)(1-(\rho v^2)^{a+1}),$$

where

$$R(a) = 1 - v + v(1-\rho v)(\rho v^2)^a.$$

Thus we find for $m \leq k$:

$$T(h,m,k) = \tag{25}$$

$$= \frac{(\alpha v^2 + \beta v + \gamma)(\rho v)^{k-m}[1 - v + v(1-\rho v)(\rho v^2)^m][1 - \rho v + \rho v(1-v)(\rho v^2)^{h-k}]}{\gamma(1-\rho v^2)(1-v)(1-\rho v)(1-(\rho v^2)^{h+1})},$$

and similary we have for $m \geq k$:

$$T(h,m,k) = \tag{26}$$

$$= \frac{(\alpha v^2 + \beta v + \gamma)v^{m-k}[1 - v + v(1-\rho v)(\rho v^2)^k][1 - \rho v + \rho v(1-v)(\rho v^2)^{h-m}]}{\gamma(1-\rho v^2)(1-v)(1-\rho v)(1-(\rho v^2)^{h+1})}.$$

In (25) and (26) the denominator unlike (13) exhibits a very simple structure, its roots being $1, \pm\rho^{-1/2}, \rho^{-1}$ and $\rho^{-1/2}e^{\theta_\nu i}, \nu = 0, 1, \ldots, h$, where $\theta_\nu = \frac{\nu\pi}{h+1}$. Using the substitution $v = \rho^{-1/2}e^{\theta i}$ and noting that there is again one complex angle, viz. $\theta = \frac{1}{2i}\log\rho$, which is due to the factor $(1-v)$, one obtains after routine calculations:

$$P(Q_h(n) = k|Q_h(0) = m) = \frac{\rho - 1}{\rho^{h+1} - 1}\rho^k + \frac{2\alpha}{h+1}\sum_{\nu=1}^{h}\frac{(\beta + 2\sqrt{\alpha\gamma}\cos\theta_\nu)^n}{1 - \beta - 2\sqrt{\alpha\gamma}\cos\theta_\nu}C_{mk}(\nu), \tag{27}$$

where

$$C_{mk}(\nu) = \left[\rho^{-\frac{m+1}{2}}\sin(m+1)\theta_\nu - \rho^{-\frac{m}{2}-1}\sin m\theta_\nu\right]\left[\rho^{\frac{k+1}{2}}\sin(k+1)\theta_\nu - \rho^{\frac{k}{2}}\sin k\theta_\nu\right],$$

which is valid even if k and (or) m are equal to h.

From (27) it may be immediately deduced that the steady state distribution is given by

$$\lim_{n\to\infty} P(Q_h(n) = k|Q_h(0) = m) = \frac{\rho - 1}{\rho^{h+1} - 1}\rho^k. \tag{28}$$

On the other hand, if we expand the terms of the summation in (27) as geometric series, then representation (5) may be applied to give an expansion in terms of generalized trinomial coefficients, which turns out to be an interesting relation linking the transition probabilities of a random walk restricted by two absorbing barriers and the steady state distribution (28):

$$P(Q_h(n) = k|Q_h(0) = m) = \tag{29}$$

$$
\begin{aligned}
&= \frac{\rho-1}{\rho^{h+1}-1}\rho^k + \\
&+ \alpha \sum_{m\geq n} [p_m(h+1, h-k, h-m) - p_m(h+1, h-k+1, h-m)] - \\
&- \gamma \sum_{m\geq n} [p_m(h+1, h-k, h-m+1) - p_m(h+1, h-k+1, h-m+1)].
\end{aligned}
$$

At this point it may be instructive to see how the spectral decomposition method works. Let V_n denote the n–th component of an eigenvector associated with an eigenvalue σ of the transition probability matrix. Then it is verified that the three term recurrence relation induced by the eigenvector equation has the general solution (see Karlin and Taylor (1981), pp. 10-18)

$$V_n(\theta) = Ae^{n\theta i} + Be^{-n\theta i},$$

and the eigenvalues will be of the following form:

$$\sigma = \beta + 2\sqrt{\alpha\gamma}\cos\theta.$$

The unknowns θ, A and B have to be determined according to the boundary conditions, which lead to the following homogenous system:

$$
\begin{aligned}
A(\sqrt{\alpha\gamma}e^{-\theta i} - \gamma) + B(\sqrt{\alpha\gamma}e^{\theta i} - \gamma) &= 0 \\
A(\sqrt{\alpha\gamma}e^{\;h+1\;\theta i} - \alpha e^{h\theta i}) + B(\sqrt{\alpha\gamma}e^{-\;h+1\;\theta i} - \alpha e^{-h\theta i}) &= 0.
\end{aligned}
$$

After elimination of A and B we obtain the following equation, which determines the angles θ:

$$\rho^{1/2}\sin(h+2)\theta - (1+\rho)\sin(h+1)\theta + \rho^{1/2}\sin h\theta = 0.$$

Since the sine function is odd, we immediately find that the roots are $\theta_\nu = \frac{\nu\pi}{h+1}, \nu = 1, 2, \ldots, h$. The root $\theta = 0$ has to be excluded, since in this case the corresponding eigenvector would be identically zero. However, it is not immediately apparent, that there is also one complex angle, viz. $\theta_0 = \frac{1}{2i}\log\rho$, which we detected by simple inspection of (25). These angles determine the eigenvalues σ:

$$\sigma_\nu = \beta + 2\sqrt{\alpha\gamma}\cos\theta_\nu \qquad (\nu = 0, 1, \ldots, h).$$

Note that $\sigma_0 = 1$, which gives rise to the first term in (27) and hence to the steady state distribution. The eigenvectors corresponding to the eigenvalues σ_ν are found by solving for the indeterminates A and B, where one of them is arbitrary. In particular, we find after some algebraic simplification that

$$\begin{aligned} V_n(\theta_0) &= \rho^{n/2} \\ V_n(\theta_\nu) &= \sin n\theta_\nu - \rho^{1/2}\sin(n+1)\theta_\nu \qquad (\nu > 0). \end{aligned}$$

These eigenvectors form an orthogonal bases. If we devide them by their euclidean norm and denote their (normalized) components by $V_n^\star(\theta_\nu)$, the transient solution is found to be

$$P(Q_h(n) = k | Q_h(0) = m) = \rho^{\frac{k-m}{2}} \sum_{\nu=0}^{h} \sigma_\nu V_m^\star(\theta_\nu) V_k^\star(\theta_\nu),$$

which after some manipulations yields (27). It may be realized that the whole procedure has to be carried out right from the beginning for sections 2.2 and 2.3. Moreover it is evident that our approach is reasonably elementary in contrast to the spectral analysis method.

3 Continuous time results

In the previous sections we have derived various results for the discrete time random walk $Z(n)$ and its associated queueing process $Q(n)$. As remarked in the introduction, it is possible to pass from discrete time to continuous time by a Poisson type limiting procedure. For this purpose we consider the time interval $(0,t)$ and split it into n subintervals of equal length $\Delta = t/n$. Set $\alpha = \lambda t/n$ and $\gamma = \mu t/n$. Now let $n \to \infty$ or, equivalently $\Delta \to 0$, while keeping λ, μ and t fixed. We expect that the finite dimensional distributions of the processes $Z(n)$ and $Q(n)$ converge to the distributions of the continuous time Markov processes $Z(t)$ and $Q(t)$, which have jump intensities λ and μ. Actually a much stronger result can be proved: it can be shown that the processes $Z(n)$ and $Q(n)$ converge in the weak sense to processes $Z(t)$ and $Q(t)$, which entails the convergence of functionals of the discrete time processes such as stopping times. For details the reader is referred to Ethier and Kurtz (1986), chapter 2, theorem 2.6, pp. 168-169. To derive the limiting forms of the finite dimensional distributions we will use the following results, which may be found in Mohanty and Panny (1990a), (1990b):

As $n \to \infty$:

$$\begin{aligned} \binom{n;\alpha,\beta,\gamma}{n+a} &= e^{-\ \lambda+\mu\ t}\rho^{a/2} I_a(2t\sqrt{\lambda\mu})(1+O(\frac{1}{n^{\epsilon}})) \qquad (\epsilon > 0), \qquad (30) \\ (\beta + 2\sqrt{\alpha\gamma}\cos\theta)^n &= e^{-\ \lambda+\mu\ t} e^{2t\sqrt{\lambda\mu}\cos\theta}(1+O(\frac{1}{n^{\epsilon}})) \qquad (\epsilon > 0), \end{aligned}$$

where $I_a(2t\sqrt{\lambda\mu})$ denotes the modified Bessel function of order a. Additionally we will require the following result:

$$\lim_{n\to\infty} \alpha \sum_{\nu \geq n} \binom{\nu;\alpha,\beta,\gamma}{\nu+a} = \lambda\rho^{a/2} \int_t^{\infty} e^{-\ \lambda+\mu\ s} I_a(2s\sqrt{\lambda\mu})\, ds. \qquad (31)$$

Equation (31) follows directly from (30). In particular we have, observing that $n = t/\Delta$:

$$\begin{aligned} \lim_{n\to\infty} \alpha \sum_{\nu \geq n} \binom{\nu;\alpha,\beta,\gamma}{\nu+a} &= \\ = \ & \lim_{n\to\infty} \lambda\Delta \sum_{i\geq 0} e^{-\ \lambda+\mu\ \ t+i\Delta}\ \rho^{a/2} I_a(2(t+i\Delta)\sqrt{\lambda\mu}) + \\ + \ & \lim_{n\to\infty} \lambda\Delta \sum_{i\geq 0} e^{-\ \lambda+\mu\ \ t+i\Delta}\ \rho^{a/2} I_a(2(t+i\Delta)\sqrt{\lambda\mu}) O(\frac{\Delta^{\epsilon}}{(t+i\Delta^2)^{\epsilon}}). \end{aligned}$$

The first sum is the Riemann sum of the integral in (31) and the second sum tends to zero as $n \to \infty$, thus (31) follows.

Using (30) we find in the case of two absorbing barriers using equation (5):

$$p_t(h,m,k) = \frac{2}{h}\rho^{\frac{k-m}{2}} e^{-\ \lambda+\mu\ t} \sum_{\nu=1}^{h-1} \sin\frac{(h-k)\nu\pi}{h} \sin\frac{(h-m)\nu\pi}{h} \exp\left[2t\sqrt{\lambda\mu}\cos\frac{\nu\pi}{h}\right], \qquad (32)$$

a result, which may be found in Neuts (1964). Alternatively we obtain from (7):

$$p_t(h,m,k) = \rho^{\frac{k-m}{2}} e^{-\ \lambda+\mu\ t} \sum_{\nu=-\infty}^{\infty} [I_{k-m-2\nu h} - I_{k+m-2\nu\ h+1}\ ], \qquad (33)$$

where I_a is an abbreviation of $I_a(2t\sqrt{\lambda\mu})$. To deal with the case of a reflecting barrier at zero and an absorbing barrier at h, we first consider formula (16). Observe that the roots θ_ν do not depend on n. Hence using (30) we find

$$P(Q(t)=k, \max_{0\leq s\leq t} Q(s) < h | Q(0)=m) = -2\rho^{\frac{k-m}{2}} e^{-\ \lambda+\mu\ t} \sum_{\nu=1}^{h} e^{2t\sqrt{\lambda\mu}\cos\theta_\nu} D_{mk}(\nu), \tag{34}$$

where

$$D_{mk}(\nu) = \frac{\sin(h-k)\theta_\nu[\sqrt{\lambda}\sin(m+1)\theta_\nu - \sqrt{\mu}\sin m\theta_\nu]}{\sqrt{\lambda}(h+1)\cos(h+1)\theta_\nu - \sqrt{\mu}\, h\cos h\theta_\nu}.$$

To derive the limiting form of (18) we note that the coefficients $d_{a,h}$ are independent of n, hence they enter into the limiting expression without change. In particular we find:

$$\begin{aligned} P(Q(t)=k, \max_{0\leq s\leq t} Q(s) < h | Q(0)=m) &= \qquad (35) \\ = \quad & e^{-\ \lambda+\mu\ t} \rho^{\frac{k-m}{2}} \sum_{a\geq 0} d_{a,h}\rho^{-a/2} \times \\ \times \quad & \Big[I_{k-m-a} - \rho^{-1/2} I_{k-m-a-1} - I_{k+m-2h-a} + \rho^{-1/2} I_{k+m-2h-a-1} \\ - \quad & I_{k+m+a-2} + \rho^{-1/2} I_{k+m+a+1} + I_{k-m+2h+a+2} - \rho^{-1/2} I_{k-m+2h+a+1} \Big]. \end{aligned}$$

It remains to show the absolute convergence of the series above. For this purpose we recall that

$$d_{a,h} = [v^a](1-v+(1-\rho v)(\rho v^2)^h)^{-1}.$$

The function $(1-v+(1-\rho v)(\rho v^2)^h)^{-1}$ is analytic in a disc around the origin and has a positive radius of convergence $R \leq 1$. Hence there is an integer A, such that

$$d_{a,h} < (\frac{1}{R} + \epsilon)^a \qquad (\epsilon > 0 \text{ and } a > A).$$

Thus setting $z = \epsilon + 1/R$ we estimate

$$\begin{aligned} \sum_{a\geq A} d_{a,h}\rho^{-a/2} I_{k+a}(2t\sqrt{\lambda\mu}) \quad &\leq \quad \sum_{a\geq A} z^a \rho^{-a/2}(t\sqrt{\lambda\mu})^{k+a} \sum_{i\geq 0} \frac{(t^2\lambda\mu)^i}{i!(i+k+a)!} \\ &\leq \quad e^{t^2\lambda\mu} \sum_{a\geq A} \frac{z^a\rho^{-a/2}(t\sqrt{\lambda\mu})^{k+a}}{a!} \\ &\to \quad 0 \text{ as } A \to \infty. \end{aligned}$$

In the case of two reflecting barriers at zero and at h we get by means of (30) the limit of (27):

$$P(Q_h(t) = k | Q_h(0) = m) = \tag{36}$$
$$= \frac{\rho - 1}{\rho^{h+1} - 1}\rho^k + 2\lambda \frac{e^{- \lambda+\mu \; t}}{h+1} \sum_{\nu=1}^{h} \frac{e^{2t\sqrt{\lambda\mu}\cos\theta_\nu}}{\lambda - 2\sqrt{\lambda\mu}\cos\theta_\nu + \mu} C_{mk}(\nu),$$

where again

$$C_{mk}(\nu) = \rho\!\left[^{-\frac{m+}{2}} \sin(m+1)\theta_\nu - \rho^{-\frac{m}{2}-1} \sin m\theta_\nu\right] \; \rho\!\left[^{\frac{k+}{2}} \sin(k+1)\theta_| nu - \rho^{\frac{k}{2}} \sin k\theta_\nu\right].$$

Formula (36) is well known, it may be found in Morse (1958) or Takacs (1962, p. 13). Finally the limit of (29) is found using (31) and expansion (7):

$$P(Q_h(t) = k | Q_h(0) = m) = \tag{37}$$
$$= \frac{\rho - 1}{\rho^{h+1} - 1}\rho^k +$$
$$+ \rho^{\frac{k-m}{2}} \sum_{\nu=-\infty}^{\infty} \int_t^\infty e^{- \lambda+\mu \; s} \times$$
$$\times \left[I_{k-m-2\nu \; h+1} + 2\sqrt{\lambda\mu}\, I_{k+m+2\nu \; h+1} - \sqrt{\lambda\mu}\, I_{k-m-1-2\nu \; h+1} \right.$$
$$\left. - \sqrt{\lambda\mu}\, I_{k-m+1-2\nu \; h+1} - \lambda I_{k+m+2+2\nu \; h+1} - \mu I_{k+m+2\nu \; h+1} \right] ds.$$

A formula similar to (37) is given by Kashyap (1965).

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