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Roppert, Josef; Derflinger, Gerhard

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Josef Roppert, Gerhard Derflinger

Institut für Statistik
Wirtschaftsuniversität Wien

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ON SOME DEFINITE INTEGRALS OF A CERTAIN CLASS OF LOGARITHMIC FUNCTIONS

J. Roppert and G. Derflinger

University of Economics and Business Administrations
Vienna, Austria

Formulae for integrals of the form

$$\int_0^1 x^{t-2}(\ln\frac{1}{x})^{n-1}\ln^2(1-x)dx,$$

$t > 0$, $n = 1, 2, \cdots$, are presented. These integrals can be expressed as finite sums involving the psi function and Riemann's zeta functions.

For some work on regression analysis[1] the one of us needed the definite integrals

$$F = \int_0^1 \frac{\ln^2 x \ln \frac{1}{1-x}}{x} dx, \quad G = \int_0^1 \frac{\ln \frac{1}{x} \ln^2 (1-x)}{x} dx.$$

Whereas it is well known[2] that $F = \frac{\pi^4}{45}$ we could not find G in the literature. Numerical integration showed that, within the accuracy of the used precedure, G equals $\frac{F}{4}$. This led us to the supposition

$$G = \frac{\pi^4}{180},$$

which we could prove later. Here we shall derive a general formula for the integrals

$$K(n,t) = \int_0^1 x^{t-2} (\ln \frac{1}{x})^{n-1} \ln^2(1-x) dx, \quad n = 1, 2, \cdots; \; t > 0. \tag{1}$$

As will be shown these integrals can be expressed as finite sums, in which the psi function and Rieman's zeta funtions occur.

We start from the beta function,

$$B(t,u) = \int_0^1 x^{t-1}(1-x)^{u-1} dx = \frac{\Gamma(t)\Gamma(u)}{\Gamma(t+u)},$$

and differentiate twice with respect to u:

$$\int_0^1 x^{t-1}(1-x)^{u-1} \ln^2(1-x) dx = \frac{\Gamma(t)\Gamma(u)}{\Gamma(t+u)} \left\{ [\psi(u) - \psi(t+u)]^2 + \varsigma(2,u) - \varsigma(2,t+u) \right\}.$$

Here $\psi(u)$ is the psi function and $\varsigma(p,u)$ is Riemann's incomplete zeta function. Setting $u = 1$ and replacing t by $t-1$ we get

$$\int_0^1 x^{t-2} \ln^2(1-x) dx = \frac{1}{t-1} \left\{ \left[\tilde{\psi}(t)\right]^2 + \varsigma(2) - \varsigma(2,t) \right\} \tag{2}$$

[1] G. Derflinger and K. Muhr. A likelihood ratio test for the inclusion of a variable in the stepwise regression analysis. Manuscript in preparation.

[2] I.S. Gradshteyn and I.M. Ryzhik. Table of Integrals, Series, and Products, 4th edition. English translation by A. Jeffrey, Academic Press, New York 1965, number 4.315, p.567.

where $\tilde{\psi}(t)$ is an abbreviation for

$$\tilde{\psi}(t) = \psi(t) - \psi(1) = \psi(t) + C$$

(C is Euler's constant). The equation (2) holds for $t > -1$; for $t \to 0$ and $t \to 1$ the limits do exist. With the abbreviation

$$A(t) = [\tilde{\psi}(t)]^2 + \varsigma(2) - \varsigma(2,t) \tag{3}$$

(2) is written as

$$\int_0^1 x^{t-2} \ln^2(1-x)dx = \frac{1}{t-1} A(t). \tag{4}$$

Differentiating $n-1$ times we get

$$\begin{aligned} K(n,t) &= (-1)^{n-1} \tfrac{d^{n-1}}{dt^{n-1}} \left[\tfrac{1}{t-1} A(t)\right] = \\ &= \tfrac{(n-1)!}{(t-1)^n} \left[A(t) + \textstyle\sum_{j=2}^{n} \tfrac{(-1)^{j-1}(t-1)^{j-1}}{(j-1)!} A^{(j-1)}(t)\right] \end{aligned} \tag{5}$$

where $A^{(j)}(t)$ means the j-th derivative of $A(t)$. In order to get these derivatives we need

$$\begin{aligned} \tfrac{d^j [\tilde{\psi}(t)]^2}{dt^j} &= \textstyle\sum_{\nu=0}^{j} \binom{j}{\nu} \tilde{\psi}^{(\nu)}(t) \tilde{\psi}^{(j-\nu)}(t) = \\ &= (-1)^j j! \left[-2\tilde{\psi}(t)\varsigma(j+1,t) + \textstyle\sum_{\nu=2}^{j} \varsigma(\nu,t)\varsigma(j+2-\nu,t)\right]. \end{aligned} \tag{6}$$

(An empty sum is zero by definition.) Then, with respect to (3), we get

$$A^{(j)}(t) = (-1)^j j! \left[-(j+1)\varsigma(j+2,t) - 2\tilde{\psi}(t)\varsigma(j+1,t) + \sum_{\nu=2}^{j} \varsigma(\nu,t)\varsigma(j+2-\nu,t)\right]. \tag{7}$$

Inserting this into (5) gives the general formula for the integrals $K(n,t)$:

$\int_0^1 x^{t-2}(\ln \frac{1}{x})^{n-1} \ln^2(1-x)dx = \frac{(n-1)!}{(t-1)^n}\{[\tilde{\psi}(t)]^2 + \varsigma(2) - \varsigma(2,t)$

$$- \sum_{j=2}^{n}(t-1)^{j-1}[j\varsigma(j+1,t) + 2\tilde{\psi}(t)\varsigma(j,t) - \sum_{\nu=2}^{j-1}\varsigma(\nu,t)\varsigma(j+1-\nu,t)]\}, \qquad (8)$$

$n = 1,2,3,\cdots;\ t > -1,\ t \neq 0,\ t \neq 1.$

Limit for $t \to 1$:

We apply l'Hospital's rule n times to (5). As $K(n,1)$ is known to be finite the first $n-1$ derivatives of the expression within the square bracket must vanish at $t=1$ in the same way as the first $n-1$ derivatives of $(t-1)^n$ do. With the help of

$$\frac{d^m}{dx^m}[x^k f(x)]_{x=0} = \binom{m}{k} k! f^{(m-k)}(0) \qquad (9)$$

we easily get

$$K(n,1) = \frac{(-1)^{n-1}}{n} A^{(n)}(1). \qquad (10)$$

Eq. (10) can also be derived by an interesting alternative way. We replace in (4) t by $t+1$ and use $A(t+1)$ as a generating function: Expanding into a McLaurin series gives

$$\begin{aligned} A(t+1) &= t\int_0^1 x^{t-1}\ln^2(1-x)dx = t\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^1 \frac{\ln^n x \ln^2(1-x)}{x}dx = \\ &\sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}(-1)^n K(n+1,1) = \sum_{n=1}^{\infty}\frac{t^n}{n!}(-1)^{n-1} n K(n,1). \end{aligned} \qquad (11)$$

(10) immediatly follows from (11). Inserting (7) with $t=1$ into (10) we obtain the formula for the case of $t=1$:

$$\int_0^1 \frac{(\ln\frac{1}{x})^{n-1}\ln^2(1-x)}{x}dx = (n-1)!\left[(n+1)\varsigma(n+2) - \sum_{\nu=2}^{n}\varsigma(\nu)\varsigma(n+2-\nu)\right]. \qquad (12)$$

Partial integration of (12) yields

$$\int_0^1 \frac{\ln\frac{1}{x}\left(\ln\frac{1}{1-x}\right)^n}{x}dx = \frac{n!}{2}\left[(n+1)\varsigma(n+2) - \sum_{\nu=2}^{n}\varsigma(\nu)\varsigma(n+2-\nu)\right]. \qquad (13)$$

Also these integrals we could not find in the literature.

Limit for $t \to 0$:

Using the identities

$$\tilde{\psi}(t) = \tilde{\psi}(t+1) - \frac{1}{t}, \quad \varsigma(\nu,t) = \varsigma(\nu,t+1) + \frac{1}{t^\nu},$$

and applying l'Hospital's rule and (9) again, one gets from (8)

$$\begin{aligned} &\int_0^1 \frac{\left(\ln\frac{1}{x}\right)^{n-1}\ln^2(1-x)}{x^2}dx = \\ &= (-1)^{n-1}(n-1)!\left\{2\varsigma(2) + \sum_{i=3}^{n+1}(-1)^i\left[(i+1)\varsigma(i) - \sum_{\nu=2}^{i-2}\varsigma(\nu)\varsigma(i-\nu)\right]\right\}. \end{aligned} \tag{14}$$

Some special cases: By setting $n = 2$ in (12) or (13) we get for the initially required integral G

$$\int_0^1 \frac{\ln\frac{1}{x}\ln^2(1-x)}{x}dx = \frac{\pi^4}{180}.$$

For $n = 3$ the equation (12) becomes

$$\int_0^1 \frac{\ln^2 x\ln^2(1-x)}{x}dx = 8\varsigma(5) - \frac{\pi^2}{3}\varsigma(3).$$

Setting $n = 2$ in (14) gives

$$\int_0^1 \frac{\ln\frac{1}{x}\ln^2(1-x)}{x^2}dx = 4\varsigma(3) - \frac{\pi^2}{3}.$$

For $t = 2$, $n = 3$ we get from (8)

$$\int_0^1 \ln^2 x\ln^2(1-x)dx = 24 - \frac{4\pi^2}{3} - 8\varsigma(3) - \frac{\pi^4}{90}.$$