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Abstract: In multiple criteria programming, a decision maker has to choose a point from the set of efficient solutions. This is usually done by some interactive procedure, where he or she moves from one efficient point to the next until an acceptable solution has been reached. It is therefore important to provide some information about the "size" of the efficient set, i.e. to know the minimum (and maximum) criterion values over the efficient set. This is a difficult problem in general. In this paper, we show that for the bicriteria problem, the problem is easy. This does not only hold for the linear bicriteria problem, but also for more general problems.

Keywords: Bicriteria programming; Nadir value.

1 Introduction

Bicriteria programming is a special instance of multiple criteria programming, where one aims at maximizing simultaneously a number $p > 1$ of objective functions. This general multiple objective problem can be formulated as follows:

$$\begin{gathered} \max z_1(x) \\ \vdots \\ \max z_p(x) \\ \text{s.t. } x \in S. \end{gathered}$$

Here $z_1, \dots, z_p : \mathbb{R}^n \to \mathbb{R}$ are the objective functions, and $S \in \mathbb{R}^n$ is the feasible set. It is well known that in multiple criteria optimization the appropriate notion of optimality is that of efficiency (or Pareto–optimality):

A point $\bar{x} \in S$ is called *efficient*, if there does not exist a point $x \in S$ which fulfills

$$\begin{aligned} z_i(x) &\geq z_i(\bar{x}) && \text{for all } i = 1, \dots, p, \text{ and} \\ z_i(x) &> z_i(\bar{x}) && \text{for at least one index } i. \end{aligned}$$

$\bar{x} \in S$ is called *weakly efficient*, if there does not exist a point $x \in S$ which fulfills

$$z_i(x) > z_i(\bar{x}) \text{ for all } i = 1, \dots, p.$$

Interchanging the role of z_1 and z_2, we similarly obtain a point e^2 by first maximizing $z_2(x)$ over S and then maximizing $z_1(x)$ over the set of all maximizers obtained.

Now evaluate the two objectives at the points e^1 and e^2. Then it is clear from the construction that every nondominated point $\bar{z}$ which is different from $z(e^1)$ fulfills

$$\bar{z}_1 < z_1(e^1).$$

Therefore, every nondominated $\bar{z}$ point must fulfill

$$\bar{z}_2 \geq z_2(e^1).$$

For a similar reasoning, every nondominated point $\bar{z} \neq z(e^2)$ must fulfill

$$\bar{z}_1 \geq z_1(e^2).$$

This shows that

$$z_1^{\text{emin}} = z_1(e^2) \qquad \text{and} \qquad z_2^{\text{emin}} = z_2(e^1). \tag{1}$$

The upper bounds on the criterion values are also provided by e^1 and e^2. It is easy to verify that

$$z_1^{\text{max}} = z_1(e^1) \qquad \text{and} \qquad z_2^{\text{max}} = z_2(e^2).$$

So in the bicriterion case, all information is contained in the two efficient points obtained by lexicographic maximization.

One can also be interested in lower bounds of the objectives over the weakly efficient set. We use the notion

$$z_i^{\text{wemin}} = \min\{z_i(x) : x \in E^W\}, \qquad i = 1, \ldots, p,$$

to denote these lower bounds. In the bicriterion case, these numbers are easy to calculate as well.

Compute a point w^1 as follows: First, maximize $z_1(x)$ over the set S, then minimize $z_2(x)$ over the set of all maximizers of the first problem. It is easy to verify that w^1 is a weakly efficient point. Interchange the role of the objectives to obtain the weakly efficient point w^2.

From the construction it follows that every weakly nondominated point z^w different from $z(w^1)$ fulfills

$$z_1^w \leq z_1(w^1).$$

All such points with $z_1^w = z_1(w^1)$ must fulfill $z_2^w > z_2(w^1)$, and all points z^w with $z_1^w < z_1(w^1)$ must fulfill $z_2^w \geq z_2(w^1)$.

Therefore, we have

$$z_2^w \geq z_2(w^1) \quad \text{for every weakly nondominated } z^w.$$

A similar reasoning can be done for the point w^2, and we conclude that

$$z_1^{\text{wemin}} = z_1(w^2) \qquad \text{and} \qquad z_2^{\text{wemin}} = z_2(w^1),$$

just in analogy to (1).

We have thus shown the following theorem:

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