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Klaus Pötzelberger

Institut für Statistik
Wirtschaftsuniversität Wien

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The quantization dimension of distributions

Klaus Pötzelberger

Institut für Statistik, WU Wien

Augasse 2-6

A-1090 Wien, Austria

Abstract

We show that the asymptotic behavior of the quantization error allows the definition of dimensions for probability distributions, the upper and the lower quantization dimension. These concepts fit into standard geometric measure theory, as the upper quantization dimension is always between the packing and the upper box-counting dimension, whereas the lower quantization dimension is between the Hausdorff and the lower box-counting dimension.

1 Introduction

The Theorem of Zador ([**9**]), which is of fundamental importance in the theory of optimal quantization, says that

$$\lim_{n\to\infty} n^{2/d}\Delta_n(\|.\|^2, P) = C_{\mathfrak{L}}(d)\left(\int\left(\frac{dP}{d\mathfrak{L}^d}\right)^{d/(2+d)} d\mathfrak{L}^d\right)^{1+2/d}, \tag{1}$$

where the quantization error is defined by

$$\Delta_n(\|.\|^2, P) = \inf_{|O|\le n}\int \min_{p\in O}\|x-p\|^2 dP(x). \tag{2}$$

$\mathfrak{L}^d$ is the d-dimensional Lebesgue measure and $dP/d\mathfrak{L}^d$ the Radon-Nikodym derivative of the absolutely continuous component of P with respect to $\mathfrak{L}^d$. $dP/d\mathfrak{L}^d = 0$ if P is singular with respect to $\mathfrak{L}^d$. $C_{\mathfrak{L}}(d)$ denotes the quantization constant, which depends on the dimension d, but not on the distribution P. (This constant is known for $d = 1$ and $d = 2$. For $d \geq 3$ only estimates exist.)

This theorem may be viewed as a first result on the relation between the asymptotic behavior of the quantization error and dimensional properties of the distribution in case the absolutely continuous component of P (with respect to Lebesgue measure) does not vanish. Analogous results have been established for special distributions which are singular with respect to Lebesgue measure. We notice that in these examples a set E with $P(E) = 1$ exists for which Hausdorff dimension and upper box-counting dimension are identical.

Results have been obtained by Graf and Luschgy for instance for the uniform distribution on the Cantor set ([**4**], [**5**]). [**10**] discusses the idea of a quantization dimension for the Cantor set on an informal level.

We define the lower quantization dimension $\underline{dim}_Q(P)$ and the upper quantization dimension $\overline{dim}_Q(P)$ as the points of discontinuity of the mappings $s \mapsto \liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P)$ and $s \mapsto \limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P)$. It is the aim of this paper to present results concerning the quantization dimensions, which allow a comparison of these concepts with popular notions of dimension of a distribution, such as Hausdorff dimension, packing dimension or box-counting dimension. It will be proved that for a suitable set E with $P(E) = 1$, $\underline{dim}_Q(P) \in [dim_{\mathcal{H}}(E), \underline{dim}_B(E)]$ and $\overline{dim}_Q(P) \in [dim_{\mathcal{P}}(E), \overline{dim}_B(E)]$. Furthermore, if E, the support of P, is a bounded s-set (i.e. $0 < \mathcal{H}^s(E) < \infty$) which additionally is of exact upper box-counting dimension $\overline{dim}_B(E) = s$ and if $P << \mathcal{H}^s$, then $\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^s, P) > 0$ and $\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^s, P) < \infty$, but in general $n^{2/s}\Delta_n(\|.\|^s, P)$ does not converge. Examples of such distributions are invariant distributions on attractors of similarity transformations satisfying a separation condition, such as the open set condition.

The Theorem of Zador as a result on the convergence of $n^{2/s}\Delta_n(\|.\|^s, P)$ may be generalized for distributions with a regular s-dimensional support. s is then necessarily integral (see [3]). (1) holds, if d is replaced by s, $\mathfrak{L}^d$ by s-dimensional Hausdorff measure $\mathcal{H}^s$ and $C_{\mathfrak{L}}(d)$ by a suitable constant. The proof of this result will be given in a forthcoming paper.

[7] provides a frame for classification and quantization. It includes approaches such as vector quantization or unsupervised learning in the theory of artificial neural nets. For a given distribution P on $\mathbb{R}^d$, a partition $\mathcal{B} = (B_1, \ldots, B_n)$ containing maximal information is sought. The information of a partition $\mathcal{B}$ is measured by

$$I^f(\mathcal{B}, P) = \int f(\mathbb{E}(X \mid \mathcal{B}))dP, \tag{3}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is an integrable convex function and $\mathbb{E}(X \mid \mathcal{B})$ is the conditional expectation of the identity $X(x) = x$, given the σ-field generated by the partition. A partition $\mathcal{B}$ is called optimal if it maximizes the information over all partitions of size n. Optimal partitions $\mathcal{B}$ match optimal quantizers $O = \{p_1, \ldots, p_n\}$ and vice versa (see [7]). In the case of $f(x) = ||x||^2$,

$B_i = \{x \mid ||x - p_i|| \leq ||x - p_k||, k = 1, \ldots, n$ and no smaller argument has this property $\}$ and $p_i = \int_{B_i} x dP/P(B_i)$. The means p_i are called prototypes in case $\mathcal{B}$ is optimal.

Notation. Let a finite Borel measure G on $\mathbb{R}^d$ and a set E be given. $G_{|E}$ and $G_{\restriction E}$ denote the restriction of G to E and the conditional distribution of G given E, i.e. $G_{|E}(A) = G(A \cap E)$ and for $0 < G(E) < \infty$, $G_{\restriction E} = G(A \cap E)/G(E)$. $\mathfrak{L}^s$ and $\mathcal{H}^s$ denote s-dimensional Lebesgue measure and s-dimensional Hausdorff measure. $B(x, r)$ denotes the open ball with center x and radius r. Let $O = \{p_1, \ldots, p_n\}$ be a set of points, not necessarily prototypes. We define

$$\Delta_n(||.||^2, O, P) = \int \min_{p \in O} ||x - p||^2 dP(x). \tag{4}$$

2 The quantization dimension for distributions of bounded support

In order to integrate the concept of the quantization dimension into geometric measure theory we restrict the results to probability measures with bounded support and to the squared norm as information function. The quantization dimension of a distribution is a dimension characteristic of the support, similar to Hausdorff, packing or box-counting dimensions. In case the support is unbounded, the asymptotic behavior of the quantization error depends additionally on regularity conditions, such as the existence of moments. The choice of the classical information function $f = \|.\|^2$ is no restriction compared to the case $f \in C^{(2)}$ with f'' regular. If f'' is not of full rank, the asymptotic behavior of $\Delta_n(f,P) := \int f dP - \max\{I^f(\mathcal{B},P) \mid |\mathcal{B}| = n\}$ simply depends not only on the geometry of the support of E, but to a large extent on features of f.

We define the lower and the upper quantization dimension of P, $\underline{dim}_Q(P)$ and $\overline{dim}_Q(P)$ as the unique real numbers s_u and $s_o \in [0,\infty]$ resp., satisfying

$$\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \begin{cases} \infty & \text{for } s < s_u, \\ 0 & \text{for } s > s_u, \end{cases}$$

$$\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \begin{cases} \infty & \text{for } s < s_o, \\ 0 & \text{for } s > s_o. \end{cases}$$

If $\underline{dim}_Q(P) = \overline{dim}_Q(P)$ we denote the common value by $dim_Q(P)$, the quantization dimension of P. (1) bounds $\overline{dim}_Q(P)$ by d.

[**3**] discusses various concepts of dimension for distributions. The (upper) Hausdorff dimension $dim^*_{\mathcal{H}}(P)$ and the packing dimension $dim^*_{\mathcal{P}}(P)$ are defined by

$$dim^*_{\mathcal{H}}(P) = \inf\{dim_{\mathcal{H}}(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}, \tag{5}$$

$$dim^*_{\mathcal{P}}(P) = \inf\{dim_{\mathcal{P}}(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}. \tag{6}$$

Analogously we define the lower and the upper box-counting dimension by

$$\underline{dim}^*_B(P) = \inf\{\underline{dim}_B(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}, \tag{7}$$

$$\overline{dim}^*_B(P) = \inf\{\overline{dim}_B(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}. \tag{8}$$

Note that for $s = dim^*_{\mathcal{H}}(P)$ (and analogously for $dim^*_{\mathcal{P}}(P)$, $\overline{dim}^*_B(P)$ and $\underline{dim}^*_B(P)$) a Borel set E exists with $P(E) = 1$ and $dim_{\mathcal{H}}(E) = s$. Upper and lower local dimensions provide an alternative definition of $dim^*_{\mathcal{H}}(P)$ and $dim^*_{\mathcal{P}}(P)$ (see [**5**]),

$$dim^*_{\mathcal{H}}(P) = \inf\{s \mid \underline{dim}_{loc}(P,x) \leq s \quad P-a.e.\}, \tag{9}$$

$$dim^*_{\mathcal{P}}(P) = \inf\{s \mid \overline{dim}_{loc}(P,x) \leq s \quad P-a.e.\}. \tag{10}$$

We recall the decomposition of measures into exact and diffuse components (see [**3**], Prop. 10.10). Only for distributions P with vanishing diffuse component exact results as Theorem 1.4-5. are possible.

THEOREM 1. *Let P be a probability distribution on $[0,1]^d$. Then*

1. $dim^*_{\mathcal{H}}(P) \leq \underline{dim}_Q(P) \leq \underline{dim}^*_B(P)$.

2. $dim^*_{\mathcal{P}}(P) \leq \overline{dim}_Q(P) \leq \overline{dim}^*_B(P)$.

3. *If $dim^*_{\mathcal{H}}(P) = \overline{dim}^*_B(P)$, then $\underline{dim}_Q(P) = \overline{dim}_Q(P) = dim^*_{\mathcal{H}}(P)$.*

4. *Let $E \subseteq [0,1]^d$ be a Borel s-set, i.e. $0 < \mathcal{H}^s(E) < \infty$, and $P << \mathcal{H}^s_{|E}$. Then*

$$\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) > 0. \tag{11}$$

5. *Let $E \subseteq [0,1]^d$ be a Borel set of exact upper box-counting dimension s with $P(E) = 1$. Then*

$$\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) < \infty. \tag{12}$$

Proof of Theorem 1. First, we prove $\overline{dim}_Q(P) \leq \overline{dim}^*_B(P)$. It is sufficient to show $\Delta_n(\|.\|^2, P) = O(1/n^{2/\alpha})$ for all $\alpha > \overline{dim}^*_B(P)$, which implies $\limsup_{n\to\infty} n^{2/\alpha}\Delta_n(\|.\|^2, P) = 0$ for $\alpha > \overline{dim}^*_B(P)$. Let $s = \overline{dim}^*_B(P)$ and $\alpha > s$. A Borel set E exists with $P(E) = 1$ and $\overline{dim}_B(E) < (s+\alpha)/2$. Let $r_0 > 0$ such that $N_K(E,r) \leq r^{-\alpha}$ for all $r \in]0, r_0]$.

$N_K(E,r)$ denotes the covering number of E by r-meshed cubes, i.e. the minimal number of r-meshed cubes required to cover E (see appendix). Let $n \geq r_0^{-\alpha}$ and $r = n^{-1/\alpha}$. Note that $\tilde{n} := N_K(E,r) \leq n$. Let $\{S_i | i \leq \tilde{n}\}$ be a cover of E with r-meshed cubes. $O_{\tilde{n}} = \{p_1, ..., p_{\tilde{n}}\}$ denotes the set of centers of the cubes. $x \in S_i$ implies $\|x - p_i\|^2 \leq r^2 d/4$ and thus

$$\Delta_n(\|.\|^2, P) \leq \Delta_{\tilde{n}}(\|.\|^2, O_{\tilde{n}}, P) \leq \sum_{i=1}^{\tilde{n}} P(S_i) r^2 d/4 = \frac{d}{4}\left(\frac{1}{n}\right)^{2/\alpha}.$$

$\underline{dim}_Q(P) \leq \underline{dim}^*_B(P)$ can be proved the same way.

If E has exact upper box-counting dimension s, then $\overline{dim}_B(E) = s$ and a constant C exists such that $N_K(E,r) \leq Cr^{-s}$. In that case we choose $r = (C/n)^{1/s}$, which leads to

$$\Delta_n(\|.\|^2, P) \leq \sum_{i=1}^{\tilde{n}} P(S_i) r^2 d/4 = \frac{d}{4}\left(\frac{C}{n}\right)^{2/s}.$$

This finishes the proof of part 5.

Before providing the proof of the remaining inequality in Theorem 1.1. (i.e. $dim^*_{\mathcal{H}}(P) \leq \underline{dim}_Q(P)$) and the exact result Theorem 1.4., we comment on the idea of the proof. Let s be the Hausdorff dimension of P, $\alpha < s$ and $(B_1, ..., B_n)$ an optimal partition with corresponding prototypes $(p_1, ..., p_n)$. Let $B(p_i, r_i)$ be a ball containing p_i, such that $P(B_i \backslash B(p_i, r_i))$ is large compared to $P(B_i)$, for instance equal to $P(B_i)/2$, and such that r_i is of order $diam(B_i)$ in magnitude. $\|x - p_i\|$ is then of order r_i for $x \in B_i \backslash B(p_i, r_i)$ and $\Delta_n(\|.\|^2, P)$ of order $\sum_{i=1}^n P(B_i) r_i^2$. (9) implies that r_i^α is at least $P(B_i)$ and thus $\Delta_n(\|.\|^2, P) \geq c \sum_{i=1}^n P(B_i)^{1+2/\alpha} \geq c/n^{2/\alpha}$ for a suitable positive constant c. However, (9) is a local result, therefore the arguments hold only on a subset $E_0 \subseteq E$ of positive probability.

Let $\alpha < s$, E a Borel set with $P(E) > 0$ and $dim_{\mathcal{H}}(E) > \alpha$. (9) implies that for P-a.e. $x \in E$ a $r_x > 0$ exists, such that for all convex sets U, $diam(U) \leq r_x$ and $x \in U$ imply $P(U) \leq diam(U)^\alpha$. But then a Borel set $E_0 \subseteq E$ with $P(E_0) > 0$ and a r_0 exist, such that for $x \in E_0$ and all convex sets U with $diam(U) \leq r_0$ and $x \in U$, $P(U) \leq diam(U)^\alpha$ holds. Let $(B_1, ..., B_n)$ be an optimal partition for $P_0 := P_{\restriction E_0}$ with conditional means $(p_1, ..., p_n)$. Let $2r_i = r_0 \wedge (P(B_i \cap E_0)/2)^{1/\alpha}$, $J_1 = \{i | r_i = (P(B_i \cap E_0)/2)^{1/\alpha}/2\}$ and

$J_2 = \{i | r_i = r_0/2\}$. Then,

$$P_0(B_i \backslash B(p_i, r_i)) \geq \frac{1}{P(E_0)}(P(B_i \cap E_0) - P(B(p_i, r_i) \cap E_0))$$

$$\geq \frac{1}{P(E_0)}(P(B_i \cap E_0) - (2r_i)^\alpha)$$

$$\geq \frac{1}{P(E_0)}(P(B_i \cap E_0) - P(B_i \cap E_0)/2) = P_0(B_i)/2.$$

The second inequality is a consequence of the definition of r_i if $B(p_i, r_i) \cap E_0 \neq \emptyset$ and holds trivially otherwise. Thus

$$n^{2/\alpha}\Delta_n(\|.\|^2, P) \geq P(E_0)n^{2/\alpha}\Delta_n(\|.\|^2, P_0)$$

$$= P(E_0)n^{2/\alpha}\sum_{i=1}^{n}\int_{B_i}\|x - p_i\|^2 dP_0(x)$$

$$\geq P(E_0)n^{2/\alpha}\sum_{i=1}^{n}\int_{B_i \backslash B(p_i, r_i)}\|x - p_i\|^2 dP_0(x)$$

$$\geq P(E_0)n^{2/\alpha}\sum_{i=1}^{n} r_i^2 P_0(B_i \backslash B(p_i, r_i))$$

$$\geq \frac{P(E_0)n^{2/\alpha}r_0^2 P_0(\cup_{i \in J_2} B_i)}{8} + \frac{P(E_0)^{1+2/\alpha}n^{2/\alpha}}{2^{3+2/\alpha}}\sum_{i \in J_1} P_0(B_i)^{1+2/\alpha}.$$

If $P_0(\cup_{i \in J_2} B_i) \geq 1/2$, the first summand is at least $P(E_0)r_0^2/16$. Otherwise, the second summand satisfies

$$\frac{P(E_0)^{1+2/\alpha}n^{2/\alpha}P_0(\cup_{j \in J_1} B_j)^{1+2/\alpha}}{2^{3+2/\alpha}}\sum_{i \in J_1}\left(\frac{P_0(B_i)}{P_0(\cup_{j \in J_1} B_j)}\right)^{1+2/\alpha}$$

$$\geq \frac{P(E_0)^{1+2/\alpha}}{2^{4+4/\alpha}}\left(\frac{n}{|J_1|}\right)^{2/\alpha} \geq \frac{P(E_0)^{1+2/\alpha}}{2^{4+4/\alpha}}.$$

This concludes the proof of $dim^*_{\mathcal{H}}(P) \leq \underline{dim}_Q(P)$.

Item 4 is the exact version of the result just proved. Note that $\overline{D}^s_c(E, x) = 1$ holds P-a.e. A Borel set $E_0 \subseteq E$ and positive constants c_1, c_2, r_0 with $r_0 < 1$ exist, such that $P(E_0) > 0$,

$$\frac{dP}{d\mathcal{H}^s_{|E}}(x) \geq c_1$$

on E_0 and

$$\mathcal{H}^s(U \cap E) \le c_2 |U|^s$$

for $x \in E_0$ and all convex sets U with $x \in U$ and $diam(U) \le r_0$. In particular, for $x \in E_0$ and convex sets U with $x \in U$ of diameter at most r_0,

$$\mathcal{H}^s(U \cap E_0) \le c_2 |U|^s.$$

We may choose $r_0 < (\mathcal{H}^s(E_0)/c_2)^{1/s}$. Again, let $(B_1, ..., B_n)$ be an optimal partition for $P_0 := P_{\restriction E_0}$ with prototypes $(p_1, ..., p_n)$. We define $c_3 = 1 - c_2 r_0^s / \mathcal{H}^s(E_0)$ and $r_i = (\mathcal{H}^s(B_i \cap E_0)(1 - c_3)/c_2)^{1/s}/2$. Then $2r_i \le r_0$, $c_3 > 0$ and

$$\begin{aligned} & P(E_0) P_0(B_i \backslash B(p_i, r_i)) \\ & \ge c_1 \mathcal{H}^s((B_i \backslash B(p_i, r_i)) \cap E_0) \\ & \ge c_1 (\mathcal{H}^s(B_i \cap E_0) - \mathcal{H}^s(B(p_i, r_i) \cap E_0)) \\ & \ge c_1 (\mathcal{H}^s(B_i \cap E_0) - c_2 (2r_i)^s) \\ & = c_1 (\mathcal{H}^s(B_i \cap E_0) - (1 - c_3) \mathcal{H}^s(B_i \cap E_0)) \\ & = c_1 c_3 \mathcal{H}^s(B_i \cap E_0). \end{aligned}$$

Thus

$$\begin{aligned} n^{2/s} \Delta_n(\|.\|^2, P) & \ge n^{2/s} P(E_0) \Delta_n(\|.\|^2, P_0) \\ & = n^{2/s} P(E_0) \sum_{i=1}^n \int_{B_i} \|x - p_i\|^2 dP_0(x) \\ & \ge n^{2/s} P(E_0) \sum_{i=1}^n \int_{B_i \backslash B(p_i, r_i)} r_i^2 dP_0(x) \\ & \ge n^{2/s} \sum_{i=1}^n r_i^2 c_1 c_3 \mathcal{H}^s(B_i \cap E_0) \\ & \ge n^{2/s} \frac{c_1 c_3}{4} ((1 - c_3)/c_2)^{2/s} \sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{1+2/s} \\ & =: c_4 n^{2/s} \sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{1+2/s}. \end{aligned}$$

Now, using Hölder's inequality, we have

$$c_4 n^{2/s} \sum_{i=1}^{n} \mathcal{H}^s(B_i \cap E_0)^{1+2/s} \geq c_4 \mathcal{H}^s(E_0)^{1+2/s}.$$

With that we have finished the proof of part 4.

To prove $dim^*_{\mathcal{P}}(P) \leq \overline{dim}_Q(P)$, let $\alpha_0 = \overline{dim}_Q(P)$ and $\alpha_1 = dim^*_{\mathcal{P}}(P)$. We assume $\alpha_0 < \alpha_1$ and deduce a contradiction. The assumption implies that for all $\alpha \in]\alpha_0, \alpha_1[$ and every sequence (n_i) with $n_i \to \infty$,

$$\lim_{i \to \infty} n_i^{2/\alpha} \Delta_{n_i}(\|.\|^2, P) = 0.$$

Let $\beta > 0$ with $\alpha_0 < 2(\alpha_1 - \beta)/(2 + \beta)$. An $\alpha_2 > \alpha_1$ exists, such that

$$\alpha_0 < \frac{2(\alpha_1 - \beta)}{2 + \beta + \alpha_2 - \alpha_1} =: \gamma. \tag{13}$$

We denote $E_0 = \{x | \overline{dim}_{loc}(P, x) > \alpha_1 - \beta/2\}$. The definition of $dim^*_{\mathcal{P}}$ implies $P(E_0) > 0$ and according to Lemma *A2.4*, $dim_{\mathcal{P}}(E_0) \geq \alpha_1 - \beta/2$ holds. For $x \in E_0$ a $\tilde{r}_x > 0$ exists, such that for all cubes S with $x \in S$ of side-length at most $|S| \leq \tilde{r}_x$, $P(S) \geq |S|^{\alpha_2}$ holds. Let (r_i) be a sequence of positive numbers with $r_i \downarrow 0$ and $E_i = \{x \in E_0 | \tilde{r}_x \geq r_i\}$. Then $E_0 = \cup E_i$. We will prove that $\overline{dim}_B(E_i) \leq \alpha_1 - \beta$. Lemma *A1.5* implies $dim_{\mathcal{P}}(E_0) \leq \alpha_1 - \beta < \alpha_1 - \beta/2$, in contradiction to $\overline{dim}_Q(P) < dim^*_{\mathcal{P}}(P)$.

Fix i and choose an arbitrary positive and decreasing nullsequence (δ_j) with $\delta_1 < r_i$. Let

$$n_j = [\frac{1}{3^d 2} N_K(E_i, \delta_j)].$$

(n_j) is unbounded. Let $O = \{p_1, ..., p_{n_j}\}$ be a set of n_j prototypes in $[0,1]^d$. The $N_K(E_i, \delta_j)$ cubes, which have nonempty intersection with E_i consist of

1. at most $n_j 3^d$ cubes that contain a point p_k, or which are neighboring such a cube,

2. the remaining cubes. Their distance to the nearest prototype is at least δ_j. There are at least $N_K(E_i, \delta_j) - n_j 3^d$ such cubes. Let $\tilde{E}$ denote the union of these cubes. Then

$$P(\tilde{E}) \geq (N_K(E_i, \delta_j) - n_j 3^d)\delta_j^{\alpha_2} \geq N_K(E_i, \delta_j)\delta_j^{\alpha_2}/2.$$

Let $\alpha \in]\alpha_0, \gamma[$. Since $\alpha > \alpha_0 = \overline{dim}_Q(P)$ a j_0 exists with $n_j^{2/\alpha}\Delta_{n_j}(\|.\|^2, P) \leq 1$ for $j \geq j_0$. But then, for $j \geq j_0$ (c_1, c_2 are constants),

$$1 \geq n_j^{2/\alpha}\Delta_{n_j}(\|.\|^2, P) \geq n_j^{2/\alpha} \int_{\tilde{E}} \min_{p\in O} \|x-p\|^2 dP(x)$$

$$\geq n_j^{2/\alpha}\delta_j^2 N_K(E_i, \delta_j)\delta_j^{\alpha_2}/2 \geq c_1 N_K(E_i, \delta_j)^{1+2/\alpha}\delta_j^{2+\alpha_2}.$$

Thus

$$N_K(E_i, \delta_j) \leq c_2 \delta_j^{-(2+\alpha_2)\alpha/(2+\alpha)}.$$

The sequence (δ_j) being arbitrary, implies

$$\overline{dim}_B(E_i) \leq (2+\alpha_2)\alpha/(2+\alpha)$$

$$\leq \frac{(2+\alpha_2)\gamma}{2+\gamma} = \frac{(2+\alpha_2)2(\alpha_1-\beta)/(2+\beta+\alpha_2-\alpha_1)}{2+2(\alpha_1-\beta)/(2+\beta+\alpha_2-\alpha_1)}$$

$$= \frac{(2+\alpha_2)(\alpha_1-\beta)}{2+\alpha_2} = \alpha_1 - \beta. \quad \square$$

REMARK 1. Theorem 1 provides upper and lower bounds on the two quantization dimensions. These bounds, which are dimensions of a set E, depend on P only as far as $P(E) = 1$ holds. However, where for instance $\overline{dim}_Q(P)$ lies in $[dim_{\mathcal{P}}^*(P), \overline{dim}_B^*(P)]$, depends on P and how it distributes its mass on E. Let for instance, E be Borel with $P(E) = 1$ and $dim_{\mathcal{P}}(E) < \overline{dim}_B(E)$. Let $\alpha \in]dim_{\mathcal{P}}(E), \overline{dim}_B(E)[$. Then $E = \cup_{i=1}^{\infty} E_i$, where $\overline{dim}_B(E_i) < \alpha$. There are constants $c_i = c_{i,\alpha}$, such that $N(E_i, r) \leq c_i r^{-\alpha}$. This sequence of constants is unbounded, since otherwise E would be of upper box-counting dimension at most α. The upper quantization dimension of P depends on the weight of sets E_i with large c_i. In case their weight is small, $\overline{dim}_Q(P)$ will be close to $dim_{\mathcal{P}}^*(P)$. Otherwise, it might be near $\overline{dim}_B^*(P)$. See Example 1.

We provide assumptions, which guarantee that $\overline{dim}_Q(P) = dim_{\mathcal{P}}^*(P)$ or $\underline{dim}_Q(P) = dim_{\mathcal{H}}^*(P)$ for distributions P on $[0,1]^d$. We define

$$\eta_{G,\alpha}(x) = \inf\{G(B(x,r))/r^\alpha \quad | \, 0 < r < d^{1/2}\}, \tag{14}$$

where G is a probability measure and $\alpha \in \mathbb{R}_+$.

The lower density of a set is a quantity closely related to the packing dimension of the set. $\eta_{P,\alpha}(x)$ depends also on large neighborhoods of x. Note that $\eta_{P,\alpha} > 0$ P-a.e. if $\alpha > dim^*_{\mathcal{P}}(P)$. In case $s = dim^*_{\mathcal{P}}(P)$, however, $\eta_{G,s}(x) > 0$ implies that $\underline{D}^s(G,.)$ is positive in x. This is the reason, why we replace $\eta_{P,s}$ by $\eta_{G,s}$ in exact results. Additionally, among random quantizers, those corresponding to P are not optimal.

Assumption *A3* controls the deviation from uniformity in the Anti-Frostman Lemma (see [**1**]). Compared to the assumptions *A1* and *A2*, which are in some sense natural, considered from the viewpoint of random quantizers, assumption *A3* has no additional and obvious interpretation.

ASSUMPTIONS. A1. *A Borel set E with $P(E) = 1$ and $dim_{\mathcal{P}}(E) = dim^*_{\mathcal{P}}(P)$ exist, such that for all $\alpha > dim^*_{\mathcal{P}}(P)$*

$$\int_E \eta_{P,\alpha}(x)^{-2/\alpha} dP(x) < \infty.$$

A2. *A Borel set E with $P(E) = 1$, $dim_{\mathcal{P}}(E) = dim^*_{\mathcal{P}}(P)$ and a distribution G exist, such that for $s = dim^*_{\mathcal{P}}(P)$,*

$$\int_E \eta_{G,s}(x)^{-2/s} dP(x) < \infty.$$

A3. *There exists a Borel set E with $P(E) = 1$ and $dim_{\mathcal{H}}(E) = dim^*_{\mathcal{H}}(P) =: s$, for all $\epsilon > 0$, $s < \alpha' < \alpha''$, there are positive constants $c(\epsilon)$ and $c(\alpha', \alpha'')$, such that for all $\delta > 0$ a measurable partition $(B_i)_{i=1}^\infty$ of E exists, satisfying*

A3.1.

$$r_i := diam(B_i) < \delta \quad \text{and } P(B_i) \geq r_i^{\alpha'} \quad \text{for all } i,$$

A3.2.

$$\sum_{i=1}^\infty P(B_i)^2 \geq c(\alpha', \alpha'') (\sum_{i=1}^\infty P(B_i)^{1+2/\alpha'})^{\alpha''/2},$$

A3.3.

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^\infty P(B_i) e^{-nP(B_i)} e^{n^{1-\epsilon} \sum_{i=1}^\infty P(B_i)^2} \leq c(\epsilon).$$

THEOREM 2. *Let P be a probability measure on $[0,1]^d$.*

1. *A1 implies $\overline{dim}_Q(P) = dim^*_{\mathcal{P}}(P)$.*

2. *A2 implies*
$$\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) < \infty, \tag{15}$$
where $s = dim^*_{\mathcal{P}}(P)$.

3. *A3 implies $\underline{dim}_Q(P) = dim^*_{\mathcal{H}}(P)$.*

The subsequent proof shows that the statements of the theorem are met also by random quantizers. A random quantizer with distribution G is a sequence $\tilde{O}_n$ of independent G-distributed random vectors, i.e. $\tilde{O}_n = \{p_1, \dots, p_n\}$ with $p_i \sim G$. $\mathbb{E}^G(\Delta_n(\|.\|^2, \tilde{O}_n, P))$, the quantization error of the random quantizer, is an upper bound of $\Delta_n(\|.\|^2, P)$.

Proof of Theorem 2. Let E be a Borel set with $P(E) = 1$ and $dim_{\mathcal{P}}(E) = dim^*_{\mathcal{P}}(P)$. Further, let $\alpha > dim^*_{\mathcal{P}}(P)$. $p_1, ..., p_n$ are independent and distributed with the law P. If $x \in E$, then
$$P(\min_{i\leq n} \|p_i - x\|^2 > t) = P(p_1 \notin B(x, t^{1/2}))^n$$
$$= (1 - P(B(x, t^{1/2}))^n \leq (1 - \eta_{P,\alpha}(x)t^{\alpha/2})^n_+$$
$$\leq e^{-n\eta_{P,\alpha}(x)t^{\alpha/2}}.$$

The quantization error of the random quantizer $\tilde{O}_n = (p_1, ..., p_n)$ is then
$$\mathbb{E}^P(\Delta_n(P, \tilde{O}_n)) = \int_0^\infty \int P(\min_{i\leq n} \|x - p_i\|^2 > t) dP(x) dt$$
$$\leq \int_0^\infty \int e^{-n\eta_{P,\alpha}(x)t^{\alpha/2}} dP(x) dt = \int \frac{\Gamma(1 + 2/\alpha)}{n^{2/\alpha}\eta_{P,\alpha}(x)^{2/\alpha}} dP(x).$$
This proves the first statement. In the proof of the second α is replaced by $s = dim^*_{\mathcal{P}}(P)$ and G is chosen as the distribution of the random quantizer.

To prove statement three, let $\alpha > s = dim_{\mathcal{H}}(E)$, where $P(E) = 1$. We choose α', α'' and ϵ with $s < \alpha' < \alpha'' < \alpha$, $\alpha'' < (1-\epsilon)\alpha$ and $\epsilon' := 1 - \epsilon - \alpha''/\alpha > 0$. Let (δ_k) be a positive nullsequence. For every k let (B^k_i) be a Borel partition of E into sets of diameter at most δ_k, which satisfy the assumption *A3* with constants $c(\epsilon)$ and $c(\alpha', \alpha'')$. Let n_k be the integer part of $(\sum_{i=1}^{\infty} P(B_i^k)^{1+2/\alpha'})^{-\alpha/2}$. (n_k) is unbounded. A sample of P-distributed random vectors of size n_k serves as prototypes $p_1, ..., p_{n_k}$. Let $x \in B_i^k$. If B_i^k contains a prototype, then $\min_j \|x - p_j\|^2 \leq r_i^2 \leq P(B_i^k)^{2/\alpha'}$. Otherwise, we replace $\min_j \|x - p_j\|^2$ by d. Then

$$\begin{aligned}
\Delta_{n_k}(\|.\|^2, P) &\leq \mathbb{E}^P(\Delta_{n_k}(\|.\|^2, \{p_1, ..., p_{n_k}\}, P)) \\
&\leq \sum_{i=1}^{\infty} P(B_i^k)\left(P(B_i^k)^{2/\alpha'} + d(1 - P(B_i^k))^{n_k}\right) \\
&\leq (n_k - 1)^{-2/\alpha} + d\sum_{i=1}^{\infty} P(B_i^k)e^{-n_k P(B_i^k)} \\
&\leq (n_k - 1)^{-2/\alpha} + dc(\epsilon)e^{-n_k^{1-\epsilon}\sum_{i=1}^{\infty} P(B_i^k)^2} \\
&\leq (n_k - 1)^{-2/\alpha} + dc(\epsilon)e^{-n_k^{1-\epsilon}c(\alpha',\alpha'')(\sum_{i=1}^{\infty} P(B_i^k)^{1+2/\alpha'})^{\alpha''/2}} \\
&\leq (n_k - 1)^{-2/\alpha} + dc(\epsilon)e^{-n_k^{1-\epsilon}c(\alpha',\alpha'')n_k^{-\alpha''/\alpha}} \\
&= (n_k - 1)^{-2/\alpha} + dc(\epsilon)e^{-n_k^{\epsilon'}c(\alpha',\alpha'')} = O(n_k^{-2/\alpha}). \quad \square
\end{aligned}$$

Theorem 3 indicates that sets, which are small in Hausdorff measure, are responsible if $\underline{dim}_Q(P) < \overline{dim}_Q(P)$. Restricting P to arbitrarily large sets leads to the equality of Hausdorff and quantization dimension. We define for $\epsilon > 0$ and $dim_{\mathcal{H}}^*(P) = s$, $\overline{dim}_Q^{\epsilon}$ by

$$\overline{dim}_Q^{\epsilon} = \inf\{\overline{dim}_Q(P_{|E}) \mid \mathcal{H}^s(E^c) < \epsilon\}. \tag{16}$$

THEOREM 3. *Let $dim_{\mathcal{H}}^*(P) = s$, E be a bounded Borel set with $P(E) = 1$, $dim_{\mathcal{H}}(E) = s$ and $dP/d\mathcal{H}_{|E}^s \neq 0$. If $\mathcal{H}^s(\{x \in E \mid \underline{D}^s(E, x) = 0\}) = 0$, then $\overline{dim}_Q^{\epsilon} = s$ for all $\epsilon > 0$.*

Proof. According to [**6**], for all $\epsilon > 0$ a set E_ϵ exists, such that $\mathcal{H}^s(E \setminus E_\epsilon) < \epsilon$ and $\overline{dim}_B(E_\epsilon) = s$. Theorem 1.2. implies $\overline{dim}_Q(P_{|E_\epsilon}) = s$. $\square$

The subsequent example demonstrates that the inequalities in Theorem 1.1. and Theorem 1.2. cannot be improved.

EXAMPLE 1. Let for $m \in \mathbb{N}$ $U_m \subseteq \mathbb{R}^2$ denote the set $\{0, 1/m, \ldots, 1 - 1/m\} \times [0, 1]$ and P_m the uniform distribution on U_m. Let $P = \sum_{m=1}^{\infty} q_m P_m$ be a mixture of such distributions. We show that weights (q_m) can be found, such that $\underline{dim}_Q(P) = 2$ and weights such that $\overline{dim}_Q(P) = 1$. Let $E = \cup_{q_m>0} U_m$. Note that $\bar{E} = [0,1]^2$ if an infinite number of weights q_m is positive, so that $dim^*_{\mathcal{P}}(P) = dim^*_{\mathcal{H}}(P) = 1$ and $\underline{dim}^*_B(P) = \overline{dim}^*_B(P) = 2$ and the lower and the upper quantization dimensions are necessarily in $[1,2]$. We have for $n = m^2$,

$$\Delta_n(\|.\|^2, P_m) = \frac{1}{12n}. \tag{17}$$

A distribution P, which is a mixture of the distributions P_m, with $\underline{dim}^*_B(P) = 2$ exists, if for alle $\epsilon > 0$ mixtures P^ϵ exist with $\underline{dim}^*_B(P^\epsilon) > 2 - \epsilon$ ($P = \sum w_k P^{1/k}$ satisfies $\underline{dim}^*_B(P) = 2$). Let $\epsilon > 0$, $s < 2 - \epsilon$ and $a \in \mathbb{N}$ with $a > s/(2-s)$. Denote by A the set $A = \{(k+1)^a \mid k \in \mathbb{N}\}$ and define q_m by

$$q_m = \begin{cases} c_1/k^2 & \text{if } m \in A, m = (k+1)^a, \\ 0 & \text{else,} \end{cases}$$

where $c_1 = 6/\pi^2$. Let $m_n = ([n^{1/2a}] + 1)^a \in A$. Thus

$$n^{2/s}\Delta_n(\|.\|^2, P) \geq n^{2/s} q_{m_n} \Delta_{m_n^2}(\|.\|^2, P_{m_n})$$

$$= \frac{c_1}{12} \frac{n^{2/s}}{[n^{1/2a}]^2([n^{1/2a}]+1)^{2a}} \geq c_2 n^{2/s-1-1/a},$$

so that $\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \infty$, implying $\underline{dim}_Q(P) \geq s > 2 - \epsilon$.

We show that for $q_m = (1/2)^m$, $\overline{dim}_Q(P) = 1$ holds. Constants c_3 and c_4 exist with $\Delta_n(\|.\|^2, P_m) \leq c_3/n$ and $\Delta_n(\|.\|^2, P_m) \leq c_4 m^2/n^2$ for all $n, m \in \mathbb{N}$. Let $i_n = [\log_2 n]$ and let $O_1, \ldots, O_{i_n}, O_{i_n+1}$ be sets of prototypes for $P_1, \ldots, P_{i_n}$ and $\sum_{m=i_n+1}^\infty (1/2)^{m-i_n} P_m$ of sizes $[n/(2i_n)], \ldots, [n/(2i_n)]$ and $n - i_n[n/(2i_n)]$, respectively. Define $O := O_1 \cup \cdots \cup O_{i_n+1}$. Thus ($c_5, c_6, c_7$ are constants),

$$\Delta_n(\|.\|^2, P) \leq \Delta_n(\|.\|^2, O, P)$$

$$\leq \sum_{m=1}^{i_n} (1/2)^m \Delta_{[n/(2i_n)]}(\|.\|^2, O_m, P_m) + (1/2)^{i_n+1} \Delta_{n-i_n[n/(2i_n)]}(\|.\|^2, O_{i_n+1}, \sum_{m=i_n+1}^\infty (1/2)^{m-i_n} P_m)$$

$$= \sum_{m=1}^{i_n} (1/2)^m \Delta_{[n/(2i_n)]}(\|.\|^2, P_m) + (1/2)^{i_n+1} \Delta_{n-i_n[n/(2i_n)]}(\|.\|^2, \sum_{m=i_n+1}^{\infty} (1/2)^{m-i_n} P_m)$$

$$\leq \sum_{m=1}^{i_n} (1/2)^m c_4 m^2/[n/(2i_n)]^2 + (1/2)^{i_n+1} c_3/(n - i_n[n/(2i_n)])$$

$$\leq c_5(i_n/n)^2 + c_6/n^2 \leq c_7(\log n/n)^2.$$

3 Appendix: Results from geometric measure theory

[1], [2], [3], [8] provide accounts of the concepts and results presented in this appendix.

1. Hausdorff and packing measure. Let $E \subseteq \mathbb{R}^d$, $\delta > 0$. A δ-cover of E is a finite or denumerably infinite family of sets $\{U_i\}$ such that $E \subseteq \cup_i U_i$ and for all i, $\text{diam}(U_i) \leq \delta$. $\text{diam}(A) := \sup\{\|x - y\| \mid x, y \in A\}$ denotes the diameter of A. For $s > 0$ let

$$\mathcal{H}^s_\delta(E) = \inf\{\sum_{i=1}^{\infty} \text{diam}(U_i)^s \mid \{U_i\} \text{ is a } \delta\text{-cover of } E\} \tag{18}$$

and

$$\mathcal{H}^s(E) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(E). \tag{19}$$

$\mathcal{H}^s(E)$ exists in $[0, \infty]$, as $\mathcal{H}^s_\delta(E)$ is increasing in δ. $\mathcal{H}^s(E)$ is the Hausdorff measure of E. $\mathcal{H}^s$ is a Borel-regular outer measure on $\mathbb{R}^d$. Borel sets are measurable. For $s = d$ $\mathcal{H}^d = c_d^{-1}\mathcal{L}^d$, where $\mathcal{L}^d$ is Lebesgue measure and $c_d = \pi^{d/2}2^{-d}/\Gamma(1 + d/2)$ is the volume of the d-dimensional ball of diameter 1.

For every set $E \subseteq \mathbb{R}^d$ a real s, the Hausdorff dimension $dim_{\mathcal{H}}(E)$ of E, exists, such that

$$\mathcal{H}^\alpha(E) = \begin{cases} \infty & \text{for } \alpha < s, \\ 0 & \text{for } \alpha > s. \end{cases}$$

An s-set is a set E such that $0 < \mathcal{H}^s(E) < \infty$, where $s = dim_{\mathcal{H}}(E)$.

Packing measure and packing dimension are defined analogously. A finite or denumerably infinite family of disjoint balls $\{B_i\}$ with centers in E is called a δ-packing of E, if for all i, $\text{diam}(B_i) \leq \delta$ holds. Define

$$\mathcal{P}^s_\delta(E) = \sup\{\sum_{i=1}^{\infty} \text{diam}(B_i)^s \mid \{B_i\} \text{ is a } \delta\text{-packing of } E\} \tag{20}$$

and

$$\mathcal{P}_0^s(E) = \lim_{\delta \to 0+} \mathcal{P}_\delta^s(E). \tag{21}$$

$\mathcal{P}_0^s$ itself is not an outer measure but allows the definition of the packing measure

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} \mathcal{P}_0^s(E_i) \mid E \subseteq \cup_i E_i\}. \tag{22}$$

The packing dimension of a set E is the discontinuity of the mapping $\alpha \mapsto \mathcal{P}^\alpha(E)$.

2. Box-counting dimension. Define the covering number $N(E,r)$ of a bounded set $E \subseteq \mathbb{R}^d$ as the minimal cardinality of a r-cover of E. The lower and the upper box-counting dimension of E are given by

$$\underline{dim}_B(E) = \liminf_{r \to 0+} \frac{\log N(E,r)}{\log(1/r)}, \tag{23}$$

$$\overline{dim}_B(E) = \limsup_{r \to 0+} \frac{\log N(E,r)}{\log(1/r)}. \tag{24}$$

A set E with $\overline{dim}_B(E) = s$ is a set of exact upper box-counting dimension, if $N(E,r) = O(r^{-s})$. Restricting the r-covers to certain classes of sets leads to the same definition of box-counting dimensions. An attractive class are covers by so-called r-meshed cubes,

$$[i_1 2^{-k}, (i_1+1)2^{-k}[\times \cdots \times [i_d 2^{-k}, (i_d+1)2^{-k}[, \tag{25}$$

with $2^{-k} \leq r$. (r-meshed cubes on the right-hand boundary of $[0,1]^d$ are modified to cover $[0,1]^d$). We denote the covering number of E by r-meshed-cubes by $N_K(E,r)$.

LEMMA A1.

1. $dim_{\mathcal{H}}(E) \leq dim_{\mathcal{P}}(E) \leq \overline{dim}_B(E)$ *and* $dim_{\mathcal{H}}(E) \leq \underline{dim}_B(E) \leq \overline{dim}_B(E)$.

2. $dim(\cup_{i=1}^k E_i) = \max_i dim(E_i)$ *for* $dim = dim_{\mathcal{H}}, dim_{\mathcal{P}}, \overline{dim}_B$.

3. $dim(\cup_{i=1}^\infty E_i) = \max_i dim(E_i)$ *for* $dim = dim_{\mathcal{H}}, dim_{\mathcal{P}}$.

4. $\overline{dim}_B(E) = \overline{dim}_B(\bar{E})$ *and* $\underline{dim}_B(E) = \underline{dim}_B(\bar{E})$.

5. *If for all $\alpha > s$ sets E_i exist, such that $E = \cup_{i=1}^{\infty} E_i$ and $\overline{dim}_B(E_i) < \alpha$, then $dim_{\mathcal{P}}(E) \le s$.*

6. *Let $dim_{\mathcal{P}}(E) = s$ and $\alpha > s$. E_i exist such that $E = \cup_{i=1}^{\infty} E_i$ and $\overline{dim}_B(E_i) < \alpha$.*

3. Local dimension. Let P be a finite Borel measure on $\mathbb{R}^d$. The lower and upper local dimension of P is defined by

$$\underline{dim}_{loc}(P,x) = \liminf_{r \to 0} \frac{\log P(B(x,r))}{\log r}, \tag{26}$$

$$\overline{dim}_{loc}(P,x) = \limsup_{r \to 0} \frac{\log P(B(x,r))}{\log r}. \tag{27}$$

LEMMA A2. Let $\alpha > 0$. Then

1. $dim_{\mathcal{H}}(\{x \mid \underline{dim}_{loc}(P,x) \le \alpha\}) \le \alpha$.

2. $A \subseteq \{x \mid \underline{dim}_{loc}(P,x) \ge \alpha\}$ *and* $P(A) > 0$ *implies* $dim_{\mathcal{H}}(A) \ge \alpha$.

3. $dim_{\mathcal{P}}(\{x \mid \overline{dim}_{loc}(P,x) \le \alpha\}) \le \alpha$.

4. $A \subseteq \{x \mid \overline{dim}_{loc}(P,x) \ge \alpha\}$ *and* $P(A) > 0$ *implies* $dim_{\mathcal{P}}(A) \ge \alpha$.

4. Density. The lower and the upper density of a Borel measure P in $x \in \mathbb{R}^d$ are defined by

$$\underline{D}^s(P,x) = \liminf_{r \to 0} \frac{P(B(x,r))}{(2r)^s}, \tag{28}$$

$$\overline{D}^s(P,x) = \limsup_{r \to 0} \frac{P(B(x,r))}{(2r)^s}. \tag{29}$$

$\overline{D}^s(E,x)$ and $\underline{D}^s(E,x)$ denote the densities of $\mathcal{H}^s_{|E}$, i.e. $\overline{D}^s(\mathcal{H}^s_{|E},x)$ and $\underline{D}^s(\mathcal{H}^s_{|E},x)$ respectively. The upper convex density (for $P = \mathcal{H}^s_{|E}$) is

$$\overline{D}^s_c(E,x) = \lim_{r \to 0} \sup\{\frac{\mathcal{H}^s(E \cap C)}{(diam(C))^s} \mid C \text{ convex}, x \in C, diam(U) \le r\}. \tag{30}$$

LEMMA A3. Let E denote a Borel set.

1. $\overline{D}^s_c(E,x) = 1$ $\mathcal{H}^s_{|E}$-*a.e.*

2. $P(E) \leq \mathcal{H}^s(E)\sup\{\overline{D}^s(P,x) \mid x \in E\}$.

3. $P(E) \geq \mathcal{H}^s(E)\inf\{\overline{D}^s(P,x) \mid x \in E\}$.

4. $P(E) \leq \mathcal{P}^s(E)\sup\{\underline{D}^s(P,x) \mid x \in E\}$.

5. $P(E) \geq \mathcal{P}^s(E)\inf\{\underline{D}^s(P,x) \mid x \in E\}$.

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