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A combination of nonparametric tests for trend in location

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Summary

A combination of some well known nonparametric tests to detect trend in location is considered. Simulation results show that the power of this combination is remarkably increased.

Keywords: Records; Inversions; Probability generating function; Monte Carlo experiments; Power; Trend;

1. Introduction

Let X_1,X_2,...,X_n be a sequence of independent random variables with continuous distribution functions $F_i(x)$. A familiar problem is to test the null hypothesis

$$H_o : F_i(x) = F(x), \quad i = 1, 2, \ldots, n,$$

i.e., the X_i's constitute a random sequence, against the alternative hypothesis

$$H_1 : F_1(x) \geq F_2(x) \geq \ldots \geq F_n(x),$$

with at least one strict inequality, i.e., the sequence of distribution functions $F_i(x)$ is stochastically increasing. This problem is usually called trend in location problem.

For this problem a number of nonparametric test procedures based on simple counting statistics is dicussed in the literature, for a review cf., e.g., Bhattacharyya (1984). Among these procedures a test based on the test statistic U_n, the number of upper records of the X's was considered by Foster and Stuart (1954) and by Brunk (1960). The random variable X_i is called an upper record, if $X_i > \max\{X_1, X_2, \ldots, X_{i-1}\}$, and analogously, X_j is called a lower record, denoted by L_n, if $X_j < \min\{X_1, X_2, \ldots, X_{j-1}\}$. Foster and Stuart suggested to reject the null hypothesis, if $U_n \geq k$ for a suitably choosen k. Simulation experiments, cf., e.g., Hackl and Katzenbeisser (1989), however, indicate that this test is less powerful than other nonparametric tests against trend, where the tests due to Mann (1945), Daniels (1950), and Cox and Stuart (1955) among others have been taken into account. The test due to Mann is based on the test statist I_n, the number of inversions among the X's, where the pair (X_i, X_j) is an inversion of the X's, if $X_i > X_j$ for $i < j$; this test may be seen as an application of Kendall's τ to the sequence (i, X_i), whereas the test due to Daniels can essentially be interpreted as an application of Spearman's rank correlation coefficient to the sequence (i, X_i).

Foster and Stuart demonstrated also that the power of the test based on U_n can be increased by considering the test statistic $U_n - L_n$. They suggested to reject the nullhypothesis, if $U_n - L_n \geq k_1$. This procedure is justified because an increasing sequence is characterized by a larger number of upper- and a small number of lower records.

On the other hand an increasing sequence also has a relatively small number of inversions. The purpose of this note is therefore to show that the power of the test due to Foster and Stuart can again be increased remarkably, if one consider a test based on the combined test statistic $T_n = U_n - L_n - I_n$. This test procedure suggests to reject the null hypothesis if $T_n \geq k_2$ for a suitably choosen k_2.

2. Distributional properties

The sequence of random variables X_1,X_2,...,X_n is under H_0 a sequence of independent and identically distributed random variables with continuous distribution function $F(x)$. To this sequence define the random variables U_n, L_n, and I_n. The supports of these random variables are the integers $\{1, 2, \ldots, n\}$, $\{1, 2, \ldots, n\}$, and $\{0, 1, \ldots, \binom{n}{2}\}$, respectively; because of technical reasons the random variable X_1 is an upper as well as a lower record of the X_i's. The joint probability generating function (pgf.) of the random variables U_n, L_n, and I_n was recently be derived by Katzenbeisser (1990): Let the pgf. be denoted by $G_n(x, y, z)$, where

$$G_n(x, y, z) = \sum_{k \geq 0} \sum_{l \geq 0} \sum_{m \geq 0} p_n(k, l, m) x^k y^l z^m,$$

with $p_n(k, l, m) = P\{U_n = k, L_n = l, I_n = m\}$, then we have

$$G_n(x, y, z) = \begin{cases} xy, & \text{if } n = 1, \\ \frac{1}{n!} xy(x + yz)(x + z + yz^2)\ldots(x + z + z^2 + \ldots + yz^{n-1}), & \text{if } n \geq 2. \end{cases}$$

This is Theorem 1 in Katzenbeisser (1990) slightly modyfied. The probability generating function for the random variable $T_n = U_n - L_n - I_n$ can now easily be derived, because the substitution of $x := s$ and $y = z := s^{-1}$ in $G_n(x, y, z)$ leads immediately to the generating function $G_{T_n}(s)$ for the random variable T_n:

$$G_{T_n}(s) = \frac{1}{n!}(s + s^{-2})(s + s^{-1} + s^{-3})...(s + s^{-1} + s^{-2} + ... + s^{-(n-2)} + s^{-n}), \quad n \geq 2$$

Expectation and variance of T_n under H_0 can be obtained from this pgf. and are given by

$$E_0\{T_n\} = -\frac{1}{2}\binom{n}{2}, \quad var_0\{T_n\} = 2\left(\frac{1}{72}\binom{n}{2}(2n+5) + n - 1\right),$$

There seems to be no easy way to derive 'closed form' expressions for the probabilities $P\{T_n = k\} := [s^k]G_{T_n}(s)$, where $k = 1-n-\binom{n}{2}, \ldots, 0, \ldots, n-1$. They can, however , for n fixed and not too large, easily be calculated using a program like MATHEMATICA (Wolfram (2nd ed. 1991)). In the following Table 1 critical values for the teststatistic T_n, sample sizes $10(2)20, 25, 30$ and $\alpha = 0.05$ and 0.01 are given.

Table 1. *Critical values for the test T_n*

	sample size							
α	10	12	14	16	18	20	25	30
0.05	-11	-19	-28	-40	-53	-68	-113	-170
0.01	-6	-13	-21	-31	-43	-57	-98	-150

Actually the tests based on T_n are slightly conservative in the sense that the true significance levels are below the nominal levels; e.g. for $n = 20(30)$ and nominal level 0.05 the true significance levels are 0.0489(0.0492).

For large n, it can be shown by means of Ljapunov's central limit theorem that the random variable T_n is asymptotically normally distributed. The pgf. $G_{T_n}(s)$ may be interpreted as the pgf. of a sum of $n-1$ independent random variables $Z_1, Z_2, ...Z_{n-1}$, i.e., $T_n \stackrel{d}{=} \sum_{i=1}^{n-1} Z_i$, where the individual Z_i's have pgf.'s

$$G_{Z_i}(s) = \frac{1}{i+1}(s + s^{-1} + s^{-2} + \ldots + s^{-(i-1)} + s^{-(i+1)}),$$

i.e., the Z_i's are uniformly distributed over the integers $\{1, -1, -2, \ldots, -(i-1), -(i+1)\}$. Expectation and variance of Z_i are given by

$$E\{Z_i\} = \mu_i = -\frac{i}{2}, \quad var\{Z_i\} = \sigma_i^2 = \frac{i(i+2)}{12} + 2.$$

Moreover, the third absolut moment of Z_i about μ_i is given by

$$E\{\|Z_i - \mu_i\|^3\} = \beta_i = \begin{cases} \frac{1}{i+1}\left(2(\frac{2+i}{2})^3 + \frac{1}{32}(i-2)^2 i^2\right), & i \quad even, \\ \frac{1}{i+1}\left(2(\frac{2+i}{2})^3 + \frac{1}{16}(i-1)^2((i-2)^2 - \frac{1}{2}(i-3)^2)\right), & i \quad odd. \end{cases}$$

Therefore $\beta_i = O(i^3)$, $\sigma_i^2 = O(i^2)$, $\sum_{i=1}^n \beta_i = O(n^4)$, and $\sum_{i=1}^n \sigma_i^2 = O(n^3)$. Following the notation in Rao (1973, p.127), let B_n and C_n be defined by $B_n = (\sum_{i=1}^n \beta_i)^{1/3}$ and $C_n = (\sum_{i=1}^n \sigma_i^2)^{1/2}$ then we have $\lim_{n\to\infty} \frac{B_n}{C_n} = 0$ because $B_n = O(n^{4/3})$ and $C_n = O(n^{3/2})$. Therefore we conclude by Ljapunov's central limit theorem that

$$\frac{T_n - E_0\{T_n\}}{\sqrt{var_0\{T_n\}}} \overset{L}{\rightarrow} Z \quad .$$

where $Z \sim N(0,1)$ distributed. Even for samller values of n the normal approximation might be sufficiently exact in many practical applications, e.g. for $n = 20$, H_0 is rejected when $T_n \geq -68$ with $p_{ex} = 0.0489$ whereas the normal approximations yields $p_{app} = 0.0552$; the corresponding values for $n = 30$ are $T_n \geq -170$, $p_{ex} = 0.0492$ and $p_{app} = 0.0527$, respectively.

3. Simulation Experiments

This section reports some results from Monte Carlo experiments performed in order to compare empirically the power of the tests based on the test statistics U_n, $U_n - L_n$ and $T_n = U_n - L_n - I_n$. Samples of size $n = 20$ were drawn from the Uniform-, Normal-, Exponential- and Logistic distribution, all in standardized form. The pseudo random numbers were generated using the corresponding NAG–routines. Under H_1 shift alternatives of the form $F_i(x) = F(x-(i-1)\Delta)$ for $i = 1, \ldots, n$ and various values of Δ have been considered. Randomization was used to ensure that all tests have a nominal significance level of 0.05; the power estimates are based on 5000 replications. The simulated values of power functions are given in Table 2 – Table 5.

Table 2. *Estimated powerfunctions for the tests based on U_n, $U_n - L_n$ and $U_n - L_n - I_n$ for the Uniform distribution.*

Δ	U_n	$U_n - L_n$	$U_n - L_n - I_n$
0.000	0.055	0.052	0.050
0.010	0.131	0.168	0.211
0.020	0.264	0.306	0.491
0.030	0.402	0.445	0.776
0.040	0.544	0.574	0.948
0.050	0.685	0.692	0.996
0.100	0.975	0.962	1.000
0.150	0.999	0.998	1.000
0.200	1.000	1.000	1.000

Table 3. *Estimated powerfunctions for the tests based on U_n, $U_n - L_n$ and $U_n - L_n - I_n$ for the Normal distribution.*

Δ	U_n	$U_n - L_n$	$U_n - L_n - I_n$
0.000	0.049	0.049	0.048
0.010	0.059	0.062	0.078
0.020	0.072	0.071	0.123
0.030	0.095	0.103	0.181
0.040	0.112	0.127	0.246
0.050	0.132	0.144	0.311
0.100	0.270	0.301	0.751
0.150	0.468	0.493	0.967
0.200	0.658	0.678	0.999

Table 4. *Estimated powerfunctions for the tests based on U_n, $U_n - L_n$ and $U_n - L_n - I_n$ for the Exponential distribution.*

Δ	U_n	$U_n - L_n$	$U_n - L_n - I_n$
0.000	0.052	0.051	0.049
0.010	0.059	0.094	0.115
0.020	0.071	0.120	0.196
0.030	0.075	0.135	0.304
0.040	0.095	0.165	0.425
0.050	0.108	0.191	0.526
0.100	0.207	0.325	0.909
0.150	0.327	0.450	0.990
0.200	0.462	0.596	0.999

Table 5. *Estimated powerfunctions for the tests based on U_n, $U_n - L_n$ and $U_n - L_n - I_n$ for the Logistic distribution.*

Δ	U_n	$U_n - L_n$	$U_n - L_n - I_n$
0.000	0.055	0.052	0.050
0.010	0.056	0.058	0.067
0.020	0.065	0.065	0.087
0.030	0.071	0.074	0.108
0.040	0.074	0.080	0.133
0.050	0.092	0.093	0.171
0.100	0.144	0.157	0.378
0.150	0.222	0.247	0.659
0.200	0.323	0.341	0.854

The results indicate that the test based on $U_n - L_n - I_n$ is remarkably more powerful in all cases considered as can be seen from Table 2. The major contribution to the power of the test based on $U_n - L_n - I_n$ is due to I_n. It is thus not surprising that the test based solely on I_n behaves in a similar way concerning the power.

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