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ABSTRACT. The transition functions for the correlated random walk with two absorbing boundaries are derived by means of a combinatorial construction which is based on Krattenthaler's Theorem for counting lattice paths with turns. Results for walks with one boundary and for unrestricted walks are presented as special cases. Finally we give an asymptotic formula, which proves to be useful for computational purposes.

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1. INTRODUCTION

In this paper we consider a particle moving randomly on the one-dimensional integer lattice. The particle starts at the origin, whenever the previous step was to the right, the particle continues to move into the same direction with probability α, it changes its direction to the left with probability $1-\alpha$. Similarly, if the last step was to the left, the particle will continue to jump to the left with probability β, and it will jump to the right with probability $1-\beta$. The lattice sites at $-s$ and t are assumed to be absorbing points. We are interested in the distribution of the particle after n jumps, subject to the constraint that no absorption took place up to time n.

More formally, the position of the particle is given by the random sum $S_n = X_1+X_2+\ldots+X_n$, where the increments X_i are determined by the states of a two-state Markov chain $X_n, n \geq 0$ with one-step transition matrix

$$\begin{array}{c} \\ +1 \\ -1 \end{array} \begin{array}{c} \begin{array}{cc} +1 & -1 \end{array} \\ \left[\begin{array}{cc} \alpha & 1-\alpha \\ 1-\beta & \beta \end{array}\right] \end{array} \tag{1}$$

In order to avoid trivialities, we assume throughout this paper that $0 < \alpha, \beta < 1$.

The random process S_n is usually referred to as *correlated random walk*, in the statistical physics literature the term *random walk with internal states* is quite common. Early work on this subject is due to Goldstein (1951), Gillis (1955) and Mohan (1955), although Weiss (1994, p. 33) claims that correlated walks have been used as models for diffusion in turbulent media as early as 1917. The relation to an interesting coin tossing game with two biased coins is discussed in Mohanty (1966) and Mohanty (1979, pp. 132). In this game the choice of the coin to be tossed depends on the outcome of the last experiment. Renshaw and Henderson (1981) indicate how correlated walks can be used to analyze the performance of a pin-ball machine. More recent work is due to Lal and Bhat (1989), Zhang (1992) and Böhm (1998). The major concern of most of the papers cited above are first-passage time problems as well as various limiting distributions, the approaches being based on difference equations and generating functions. Exceptions are the papers by Lal and Bhat (1989), where matrix methods are used and Mohanty (1966), who uses a combinatorial argument.

It is the combinatorial argument, as has been demonstrated very clearly by Feller (1968, chapter III), which allows the most elegant and simplest solution of one- and two-boundary absorption problems for simple random walks. The elegance is entirely due to André 's reflection principle, which leads directly to a representation of the transition functions of the restricted random walk in terms of those of the unrestricted walk. Gessel and Zeilberger (1992) have shown that this is a general pattern, holding also for walks in higher dimensional spaces restricted to Weyl chambers, provided certain regularity conditions regarding the restricting hyper planes and the allowed steps are satisfied. So one can go very far, using the simple device of reflections only. Does a similar argument work also in the present context of correlated walks? Some (mental) reflection immediately suggests that the affirmative answer has to be: no. This is easy to see. In the case of a simple random walk the distribution of a sample path is determined by the number of steps of type $+1$ or -1. Not so for correlated walks. Here it is the number of *subsequences of length 2* of steps, which we will call *turns*, or simply the number of transitions of various types of the governing Markov chain, characterized by (1). Whenever we reflect parts of a sample path at a boundary, the number of turns changes in a complicated way. The situation becomes intractable, when there are two boundaries and one has to apply reflections repeatedly.

However, only recently Krattenthaler (1997) has discovered a beautiful formula which counts the number of lattice paths between two parallel lines and a fixed number of north-east turns. In the next section we will first show that the probability distribution of S_n is completely specified by the initial distribution and the number of $(+1,-1)$- turns *or* the number of $(-1,+1)$-turns of the governing Markov chain. By a simple decomposition Krattenthaler's formula becomes applicable and in this way we derive the transition functions of S_n in presence of one and two absorbing boundaries as well as in the unrestricted case. Surprisingly it will turn out that there is a strong structural similarity between our results and those for simple random walks. Finally, in section 3 we present a useful asymptotic formula.

2. About paths with turns

The starting point for the construction of a combinatorial solution for correlated random walks with absorbing boundaries is Krattenthaler's theorem (Krattenthaler (1997)), which we cite here for future reference. In this theorem the term *lattice path* has to be understood in the sense of *minimal lattice path*, i.e. paths defined on the two-dimensional integer lattice, consisting two types of steps only: $s_1 = (1,0)$ and $s_2 = (0,1)$. A *north-east (NE) turn* is defined as a s_1- step followed by a s_2- step.

Theorem 2.1 (Krattenthaler). *Let* $t > s$ *be integers and let* $c + t \geq d \geq c + s$ *and* $v + t \geq w \geq v + s$. *Then the number of path from the point* (c, d) *to* (v, w) *which do not cross the lines* $y = x + t$ *and* $y = x + s$, *having exactly* ℓ NE-*turns is given by*

$$\text{(2)} \qquad L\left((c,d) \to (v,w),\ x+t \geq y \geq x+s, \textit{number of } \mathrm{NE}\textit{-turns} = \ell\right) = \\ = \sum_{k\in\mathbb{Z}} \left[\binom{v-c-k(t-s)}{\ell+k} \binom{w-d+k(t-s)}{\ell-k} - \right. \\ \left. - \binom{v-d-k(t-s)+s-1}{\ell+k} \binom{w-c+k(t-s)-s+1}{\ell-k} \right]$$

The proof of this theorem is based on an ingenuine representation of paths with turns by certain two-rowed integer arrays, which allows the construction of bijections in a manner very similar to the classical principle of repeated reflections, as it is used to prove the corresponding two-boundary result for paths representing simple random walks with independent increments.

Remark: In (2) and all subsequent formulas we assume binomial coefficients to be zero whenever the upper parameter becomes negative.

In what follows it will be more convenient to formulate our results in terms of lattice paths in the sense of Feller (1967, pp. 68), i.e. paths which consist of $(1,1)$- and $(1,-1)$ steps, which we will call U- and D-steps. Note that Krattenthaler's paths are translated into Feller-paths by a simple rotation of the coordinate system.

The paths we consider have $n + 1$ steps including a zero-th step in order to take care of the initial condition $X_0 = \pm 1$. In particular we define an UD-path (up-down) as a path which terminates with a -1 step, the zero-th step is $+1$. Similarly we define DD-, DU- and UU-paths.

The subsequence of two steps of type (U, D) is called a UD-*turn*, the subsequence (D, U) a DU-*turn*.

In order to specify the probability distribution of a correlated random walk, we need information about the number of transitions of the governing Markov chain. The various transition counts are given by the following

Lemma 2.2. *For the four different types of paths with* k UD*-turns,* m *steps of type* U *and* n *steps of type* D*, the transition counts of the governing Markov chain are given by:*

UU-*paths*

	U	D	
U	$m-k$	k	m
D	k	$n-k$	n

DD-*paths*

	U	D	
U	$m-k$	k	m
D	k	$n-k$	n

UD-*paths*

	U	D	
U	$m-k+1$	k	m
D	$k-1$	$n-k$	n

DU-*paths*

	U	D	
U	$m-k-1$	k	m
D	$k+1$	$n-k$	n

These transition counts are independent of further properties of the paths. They hold for unrestricted paths as well as for paths which are not allowed to touch one or two absorbing boundaries.

Proof: Let n_{UD}, n_{DU}, n_{UU} and n_{DD} denote the transition counts of the governing chain. For a UU-path necessarily

$$n_{UD} = n_{DU} = k, \qquad \text{(say)}.$$

The number of U steps equals the number of visits to state $+1$ of the governing chain, therefore

$$m = n_{UU} + n_{DU} \Rightarrow n_{UU} = m - k.$$

Similarly we get $n_{DD} = n - k$. Exactly the same argument holds for DD-paths. Consider now UD-paths. For such paths the number of DU-turns must be one less than the number of UD-turns. Thus if $n_{UD} = k$, then $n_{DU} = k - 1$. Since $m = n_{UU} + n_{DU}$, it follows that $n_{UU} = m - k + 1$. Similarly $n_{DD} = n - k$. Again the same argument holds for DU-paths, this time, however, $n_{UD} = n_{DU} - 1$. □

Let us define the event

$$A_{s,t} = \{-s < S_i < t, i = 1 \dots, n; s, t \in \mathbb{N}\}.$$

We will allow also either t or s or both to become infinitely large, in order to cover the situation with one lower or upper absorbing boundary as well as the case of an unrestricted correlated walk.

Now let $N^*_{UD}(a, b, A_{s,t}|k)$ denote the number of paths with k UD-turns, a steps of type U, b steps of type D and property $A_{s,t}$. There is no zero-th step. Similarly define $N^*_{DU}(a, b, A_{s,t}|k)$ as the number of paths with the same properties but exactly k DU-turns. These counting functions will serve as a link to Krattenthaler's theorem.

We will also require the following notation:

1. $N_{DU}(a, b, A_{s,t}|k)$ and $N_{DD}(a, b, A_{s,t}|k)$ denote the number of DU-paths (DD-paths) with a steps of type U, b steps of type D and k UD-turns;
2. $M_{UD}(a, b, A_{s,t}|k)$ and $M_{UU}(a, b, A_{s,t}|k)$ denote the number of UD-paths (UU-paths) with a steps of type U, b steps of type D and k DU-turns;

In each case it is assumed that the paths to be enumerated share property $A_{s,t}$. Observe that the enumerating functions N_{DU} and N_{DD} count paths with UD-turns, whereas the functions M_{UU} and M_{UD} count paths with respect to DU-turns.

The relation between these functions and N^*_{UD} and N^*_{DU} is given in

Lemma 2.3.

$$N_{DU}(a, b, A_{s,t}|k) = N^*_{UD}(a-1, b, A_{s,t}|k) \tag{3}$$

$$N_{DD}(a, b, A_{s,t}|k) = N^*_{UD}(a, b, A_{s,t}|k) - N^*_{UD}(a-1, b, A_{s,t}|k) \tag{4}$$

$$M_{UD}(a, b, A_{s,t}|k) = N^*_{DU}(a, b-1, A_{s,t}|k) \tag{5}$$

$$M_{UU}(a, b, A_{s,t}|k) = N^*_{DU}(a, b, A_{s,t}|k) - N^*_{DU}(a, b-1, A_{s,t}|k). \tag{6}$$

Proof: Consider a particular path counted by N_{DU}. If we delete the zero-th and the last step, then the number of UD-turns does not change and we are left with paths starting at the origin with $a-1$ steps of type $+1$, b steps of type -1 and still k UD-turns. Their number is given by the right hand side of (3).

To prove (4) observe that

$$N_{DU}(a, b, A_{s,t}|k) + N_{DD}(a, b, A_{s,t}|k)$$

equals the total number of paths starting at the origin, having k UD-turns, their number being $N^*_{UD}(a, b, A_{s,t}|k)$. This proves (4).

Next consider UD-paths, this time with k DU-turns. If we delete the zero-th and the last step, again the number of DU-turns remains unchanged and we have a path starting at the origin with a steps of type $+1$, $b-1$ steps of type -1 with k DU-turns and property $A_{s,t}$. Their number equals $N^*_{DU}(a, b-1, A_{s,t}|k)$.

Since

$$M_{UD}(a, b, A_{s,t}|k) + M_{UU}(a, b, A_{s,t}|k) = N^*_{DU}(a, b, A_{s,t}|k)$$

(6) follows from (5) and the above equation. □

We are now able to formulate the transition functions of the correlated random walk with absorbing boundaries in terms of the functions N^*_{UD} and N^*_{DU} while keeping track of the transition counts of the governing Markov chain via Lemma 2.2 and Lemma 2.3:

Theorem 2.4. *Let $A_{s,t}$ be as defined above, s and/or t being possibly infinite, and let*

$$P_\pm(n, k, A_{s,t}) = P(S_n = k, A_{s,t}|X_0 = \pm 1).$$

Then

(7)

$$P_+(n, k, A_{s,t}) = \alpha^{\frac{n+k}{2}} \beta^{\frac{n-k}{2}} \sum_{\ell \geq 0} \left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^\ell N^*_{DU}\left(\frac{n+k}{2}, \frac{n-k}{2}, A_{s,t}|\ell\right) -$$

$$- (\alpha+\beta-1)\alpha^{\frac{n+k}{2}} \beta^{\frac{n-k-2}{2}} \sum_{\ell \geq 0} \left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^\ell N^*_{DU}\left(\frac{n+k}{2}, \frac{n-k-2}{2}, A_{s,t}|\ell\right)$$

and

(8)

$$P_-(n, k, A_{s,t}) = \alpha^{\frac{n+k}{2}} \beta^{\frac{n-k}{2}} \sum_{\ell \geq 0} \left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^\ell N^*_{UD}\left(\frac{n+k}{2}, \frac{n-k}{2}, A_{s,t}|\ell\right) -$$

$$- (\alpha+\beta-1)\alpha^{\frac{n+k-2}{2}} \beta^{\frac{n-k}{2}} \sum_{\ell \geq 0} \left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^\ell N^*_{UD}\left(\frac{n+k-2}{2}, \frac{n-k}{2}, A_{s,t}|\ell\right)$$

Proof: Consider first $P_+(n, k, A_{s,t})$. The paths corresponding to this probability belong to the classes UD and UU. By Lemma 2.2 a UU-path with ℓ DU-turns leading from the origin to height k after n steps has probability

$$\alpha^{\frac{n+k}{2}}(1-\alpha)^\ell(1-\beta)^\ell\beta^{\frac{n-k}{2}}.$$

For a UD-path with the same properties we find for its probability

$$\alpha^{\frac{n+k}{2}-1}(1-\alpha)^\ell(1-\beta)^{\ell+1}\beta^{\frac{n-k}{2}}.$$

Using the representations (5) and (6) we get (7). Formula (8) is proved in exactly the same way. □

What remains to be done is it to determine the functions N^*_{UD} and N^*_{DU}. This task is accomplished most easily by Krattenthaler's theorem.

In (2) put $c = d = 0$ and $v = a, w = b$ and shift the boundaries one unit away from the origin. Then for integers $t, s > 0$ we have immediately:

$$N^*_{UD}(a,b,A_{s,t}|\ell) = \sum_{k\in\mathbb{Z}}\left[\binom{b-k(s+t-2)}{\ell+k}\binom{a+k(s+t-2)}{\ell-k} - \right. \\ \left. - \binom{b-k(s+t-2)-s}{\ell+k}\binom{a+k(s+t-2)+s}{\ell-k}\right] \tag{9}$$

The corresponding result for paths with DU-turns is found by simply reflecting the random walk paths at the x-axis and replacing $s \leftrightarrow t$ and $a \leftrightarrow b$. This yields:

$$N^*_{DU}(a,b,A_{s,t}|\ell) = \sum_{k\in\mathbb{Z}}\left[\binom{a-k(s+t-s)}{\ell+k}\binom{b+k(s+t-2)}{\ell-k} - \right. \\ \left. - \binom{a-k(s+t-2)-t}{\ell+k}\binom{b+k(s+t-2)+t}{\ell-k}\right] \tag{10}$$

Using these results in (7) and (8) we obtain finally

Theorem 2.5. *Let* $\lambda = (1-\alpha)/\beta$ *and* $\mu = (1-\beta)/\alpha$. *Then*

$$P_+(n,k,A_{s,t}) = \sum_{m\in\mathbb{Z}} [g_m(n,k,s+t-2,0,\lambda) - g_m(n,k,s+t-2,t,\lambda)] \tag{11}$$

and

$$P_-(n,k,A_{s,t}) = \sum_{m\in\mathbb{Z}} [g_m(n,-k,s+t-2,0,\mu) - g_m(n,-k,s+t-2,s,\mu)] \tag{12}$$

where the functions g_m *are defined by*

(13)

$$g_m(n,k,r,t,z) = \\ = \alpha^{\frac{n+k}{2}}\beta^{\frac{n-k}{2}}\sum_{\ell\geq 0}(\lambda\mu)^\ell\binom{\frac{n+k}{2}-mr-t}{\ell+m}\binom{\frac{n-k}{2}+mr+t}{\ell-m}\left(z+(1-z)\frac{2(\ell-m)}{n-k+2mr+2t}\right).$$

Proof: Insert simply (10) in (7) and (9) in (8). Using the fact that

$$\binom{c}{d} - z\binom{c-1}{d} = \binom{c}{d}(1-z+z\frac{d}{c}) \tag{14}$$

we get after some trivial simplifications (11) and (12). □

If we let $s, t \to \infty$ in (11) and (12), we immediately get the following additional results for the one-boundary case and for unrestricted walks (see the remark after Theorem 2.1):

Corollary 2.6. *For one upper absorbing boundary at* $t > 0$*:*

$$P_+(n,k,A_{\infty,t}) = g_0(n,k,0,0,\lambda) - g_0(n,k,0,t,\lambda), \tag{15}$$

$$P_-(n,k,A_{\infty,t}) = \left(\frac{\alpha}{\beta}\right)^k (g_0(n,-k,0,0,\mu) - g_{-1}(n,-k,t-2,0,\mu)),. \tag{16}$$

For one lower absorbing boundary at $-s, s > 0$*;*

$$P_+(n,k,A_{s,\infty}) = g_0(n,k,0,0,\lambda) - g_{-1}(n,k,s-2,0,\lambda) \tag{17}$$

$$P_-(n,k,A_{s,\infty}) = \left(\frac{\alpha}{\beta}\right)^k (g_0(n,-k,0,0,\mu) - g_0(n,-k,0,s,\mu)). \tag{18}$$

For the unrestricted case:

$$P_+(n,k,A_{\infty,\infty}) = g_0(n,k,0,0,\lambda), \tag{19}$$

$$P_-(n,k,A_{\infty,\infty}) = \left(\frac{\alpha}{\beta}\right)^k g_0(n,-k,0,0,\mu). \tag{20}$$

Remark. It is interesting to observe that (15) as well as (18) may be expressed solely in terms of the transition functions of the unrestricted walk (19) and (20), as it is the case for walks with independent increments. However, this analogy breaks down in the other cases due to the asymmetry in the initial conditions. Still we may expect that this simple picture is recoverable when n grows large.

3. An asymptotic formula

The building blocks of our formulas are the functions g_m, which may be calculated by means of (13). But some reflection and numerical experimentation will convince us that (13) is not very well suited for this purpose when n becomes large. The product of binomial coefficients grows very rapidly, not to talk about the factor $(\lambda\mu)^\ell$, because $\lambda\mu$ may be greater than one. Therefore it will be convenient to have an asymptotic approximation of the functions g_m which in turn will give rise to asymptotic approximations for all the cases discussed above.

In what follows we will derive such an asymptotic formula, however, we will restrict our interest to the symmetric case $\alpha = \beta$.

First of all, note that $g_m(n,k,r,t,z)$ may be split into two summations according to (14):

$$g_m(n,k,r,t,z) = \alpha^n \Big[\sum_\ell \delta^{2\ell} \binom{\frac{n+k}{2} - mr - t}{\ell + m} \binom{\frac{n-k}{2} + mr + t}{\ell - m} - - (1-z) \sum_\ell \delta^{2\ell} \binom{\frac{n+k}{2} - mr - t}{\ell + m} \binom{\frac{n-k}{2} + mr + t - 1}{\ell - m} \Big], \tag{21}$$

where $\delta = (1-\alpha)/\alpha$. Thus it is sufficient to find an approximation for the sum

$$f_m(n,k,x,y,z) = \alpha^n \sum_\ell \delta^\ell \binom{\frac{n+k}{2} - x}{\ell + m} \binom{\frac{n-k}{2} + y}{\ell - m}. \tag{22}$$

Here $x = mr + t$ and $y = mr + t$ for the first summation and $y = mr + t - 1$ for the second summation in (21).

The sum (22) may be regarded as a *Hadamard product* of the series

$$\sum_\ell \delta^\ell \binom{\frac{n+k}{2} - x}{\ell + m} \quad \text{and} \quad \sum_\ell \delta^\ell \binom{\frac{n+k}{2} + y}{\ell - m}$$

neglecting for the moment the leading α^n. According to results of Brag (1999) an integral representation is therefore readily available:

$$f_m(n,k,x,y,z) = \frac{\alpha^n}{2\pi} \int_{-\pi}^{\pi} (1 + \delta e^{\theta i})^{\frac{n+k}{2} - x} (1 + \delta e^{-\theta i})^{\frac{n-k}{2} - y} e^{-2m\theta i}\, d\theta. \tag{23}$$

By a standard argument used in the proof of Laplace's method for integrals (see Henrici (1977), pp. 409) it is not difficult to show that the integrand has a sharp maximum at zero and that it is exponentially small outside the interval $(-n^{-1/2+\epsilon}, n^{-1/2+\epsilon})$, for some $\epsilon > 0$. Therefore putting $\theta = t/\sqrt{n}$ and extending the range of integration to the whole real line, we are left with

(24)

$$f_m(n,k,x,y,z) = \frac{\alpha^n}{2\pi\sqrt{n}} \int_{-\infty}^{\infty} (1 + \delta e^{ti/\sqrt{n}})^{\frac{n+k}{2} - x} (1 + \delta e^{-ti/\sqrt{n}})^{\frac{n-k}{2} + y} e^{-2mti/\sqrt{n}}\, dt + O(e^{-n^\epsilon})$$

In what follows we will omit the exponential error term in our formulas and ask the reader to interpret equality signs appropriately.

The Taylor expansion of $\ln((1 + \delta e^{ti/\sqrt{n}})^{\frac{n+k}{2} - x} (1 + \delta e^{-ti/\sqrt{n}})^{\frac{n-k}{2} + y})$ yields:

$$c_0 + c_1 ti + c_2 t^2 + c_3 t^3 i + c_4 t^4 + O(1/n^2), \ ,$$

where

$$c_0 = (n-x+y)\ln(1+\delta) = -(n-x+y)\ln\alpha \tag{25}$$
$$c_1 = -\frac{\delta(x+y-k)}{(1+\delta)n^{1/2}} = -\frac{(1-\alpha)(x+y-k)}{n^{1/2}}$$
$$c_2 = -\frac{\delta(n-x+y)}{2n(1+\delta)^2} = -\frac{\alpha(1-\alpha)(n-x+y)}{2n}$$
$$c_3 = -\frac{\delta(\delta-1)(x+y-k)}{6(1+\delta)^3 n^{3/2}} = \frac{\alpha(1-\alpha)(2\alpha-1)(x+y-k)}{6n^{3/2}}$$
$$c_4 = \frac{\delta(n-x+y)(\delta^2-4\delta+1)}{24(1+\delta)^4 n^2} = \frac{\alpha(1-\alpha)(n-x+y)(6\alpha^2-6\alpha+1)}{24n^2}.$$

Thus we may write after separating real and imaginary parts

$$f_m(n,k,x,y,z) = \frac{\alpha^{x-y}}{2\pi\sqrt{n}}\int_{-\infty}^{\infty} e^{c_2t^2}\cos\left(\frac{c_1\sqrt{n}-2m}{\sqrt{n}}t\right)(1+c_4t^4+O(1/n^2))dt$$
$$-\frac{\alpha^{x-y}}{2\pi\sqrt{n}}\int_{-\infty}^{\infty} e^{c_2t^2}\sin\left(\frac{c_1\sqrt{n}-2m}{\sqrt{n}}t\right)(c_3t^3+O(1/n^2))dt$$

These integrals are known (see Gradshteyn, Rhyzhik (1980), p. 496), they can be evaluated in terms of Hermite polynomials and exponential functions. Using these known results, we find that if we neglect the second integral this induces only a relative error of $O(1/n^2)$. Thus keeping only first order terms in n, we obtain:

$$f_m(n,k,x,y,z) = \frac{\alpha^{x-y}}{\sqrt{2\pi\alpha(1-\alpha)(n-x+y)}}\exp\left[-\frac{(2m+(1-\alpha)(x+y-k))^2}{2\alpha(1-\alpha)(n-x+y)}\right]\times \tag{26}$$
$$\times\left(1+\frac{6\alpha^2-6\alpha+1}{8\alpha(1-\alpha)n}+O(1/n^2)\right)$$

Using (26) in (21), putting $x = mr+t$ and $y = mr+t$ ($y = mr+t-1$), pulling out the common exponential factor and keeping track of terms of order $1/n$, we get finally after some simplification the following theorem:

Theorem 3.1. *As* $n \to \infty$ *and while keeping* k, m, r, t *fixed:*

(27)

$$g_m(n,k,r,t,z) = \sqrt{\frac{2}{n\pi}}\sqrt{\frac{1-\alpha}{\alpha}}\exp\left[-\frac{(2m+(1-\alpha)(2mr+2t-k))^2}{2\alpha(1-\alpha)n}\right]\times$$
$$\times\left(1-\frac{(2\alpha-1)(2m+(1-\alpha)(2mr+2t-k))}{2\alpha(1-\alpha)n}+\frac{2\alpha-2\alpha^2-1}{8\alpha(1-\alpha)n}+O(1/n^2)\right).$$

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