On Zero avoiding Transition Probabilities of an r-node Tandem Queue - a Combinatorial Approach

Böhm, Walter; Jain, J. L.; Mohanty, Sri Gopal

Published: 01/01/1992

Document Version
Publisher's PDF, also known as Version of record

On Zero avoiding Transition Probabilities of an r-node Tandem Queue - a Combinatorial Approach

Walter Böhm, J.L. Jain, Sri Gopal Mohanty

Institut für Statistik
Wirtschaftsuniversität Wien

Forschungsberichte

Bericht 26
1992

http://statmath.wu-wien.ac.at/

On zero avoiding transition probabilities of an r-node tandem queue- a combinatorial approach.

Walter Böhm
Vienna

J. L. Jain
New Delhi

S.G. Mohanty
Hamilton

Abstract

In this paper we present a simple combinatorial approach for the derivation of zero avoiding transition probabilities in a Markovian r- node series Jackson network. The method we propose offers two advantages: first, it is conceptually simple because it is based on transition counts between the nodes and does not require a tensor representation of the network. Second, the method provides us with a very efficient technique for numerical computation of zero avoiding transition probabilities.

Keywords: tandem queue, zero avoiding transition probabilities, combinatorial methods, lattice paths.

In this note we consider an r- node tandem queueing model, $r > 1$, subject to the following assumptions:

- customers arrive at the first node according to a Poisson process with rate λ
- the service times at node $i, 0 \leq i \leq r$ are exponentially distributed with means $1/\mu_i$
- initially there are m_i customers waiting at node i

Let $Q_i(t)$ denote the number of customers at node i (including the one being served) at time t and let $T_i = \inf\{t : Q_i(t) = 0\}$. For notational convenience

In 1987 W. Massey published a paper in which he derived by operator analytic methods various representations of the zero avoiding transition probabilities probabilities

$$q_t(\mathbf{m},\mathbf{k}) = P(\mathbf{Q}(t) = \mathbf{k}, \mathbf{T} > t | \mathbf{Q}(0) = \mathbf{m}), \tag{1}$$

in terms of lattice Bessel functions of rank r. In a subsequent paper Baccelli and Massey (1990) have shown that the distribution of $\mathbf{Q}(t)$ without any restriction on the stopping times T_i is closely related to the probabilities $q_t(\mathbf{m},\mathbf{k})$. So these probabilities play an important role in the study of tandem queues. Unfortunately it turns out the the representation of $q_t(\mathbf{m},\mathbf{k})$ in terms of lattice Bessel functions is not very well suited for numerical computation, since the numerical determination of $q_t(\mathbf{m},\mathbf{k})$ requires the evaluation of $(r+1)!$ lattice Bessel functions, which is a rather time consuming task even for moderate values of r.

In this paper we give an alternative representation of $q_t(\mathbf{m},\mathbf{k})$ by purely combinatorial arguments which has additionally the appealing property of giving rise to an efficient numerical technique. For this purpose we will need a combinatorial theorem which may be found for instance in Kreweras (1965, p. 33)

Theorem 1 *Let $N(r;\mathbf{a},\mathbf{b})$ denote the number of k-dimensional lattice paths from the point $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ to the point $\mathbf{b} = (b_1, b_2, \ldots, b_r)$ such that every lattice point $(x_1, x_2, \ldots, x_r)$ on the paths satisfies the condition $x_1 \geq x_2 \geq \ldots \geq x_r$. Then*

$$N(r;\mathbf{a},\mathbf{b}) = \left(\sum_{\nu=1}^{r}(b_\nu - a_\nu)\right)! \|c_{ij}\|_{r\times r}, \tag{2}$$

where $\|c_{ij}\|_{r\times r}$ is the $r \times r$ determinant with (i,j)–th element given by

$$c_{ij} = \frac{1}{(b_i - a_j - i + j)} \qquad i,j = 1,2,\ldots,r$$

Kreweras' original result is in terms of Young chains, when converted into lattice paths has the above representation.

Let the point of time where there is either an arrival or a service completion be called an instant. Denote by α_{0j} and $\alpha_{ij}, i = 1,2,\ldots,r$ the number of customers arrived at instant j at the first node and the number of units served at the i–th node further that the total number of instants up to time t is equal to N. The event corresponding to $q_t(\mathbf{m}, \mathbf{k})$, viz.

$$\{\mathbf{Q}(t) = \mathbf{k}, \mathbf{T} > t | \mathbf{Q}(0) = \mathbf{m}\}$$

can occur only if at any node i and for all instants $\nu = 1, 2, \ldots, N$ the number of departures is strictly less than the number of arrivals plus the number of customers waiting at time zero. In other words

$$m_i + \sum_{\ell=1}^{\nu} \alpha_{i-1,\ell} > \sum_{\ell=1}^{\nu} \alpha_{i,\ell},$$

or equivalently

$$m_i - 1 + \sum_{\ell=1}^{\nu} \alpha_{i-1,\ell} \;\; \geq \;\; \sum_{\ell=1}^{\nu} \alpha_{i,\ell} \qquad \text{for} \quad i = 1, 2, \ldots, r; \nu = 1, 2, \ldots, N \tag{3}$$

Let us now assume that in $(0, t)$ in total n customers arrive at node 1. Thus

$$\sum_{\ell=1}^{N} \alpha_{0,\ell} = n. \tag{4}$$

Since at time t the number of customers waiting is equal to k_i, we have

$$m_i + \sum_{\ell=1}^{N} \alpha_{i-1,\ell} - \sum_{\ell=1}^{N} \alpha_{i,\ell} = k_i,$$

which together with (4) yields

$$\sum_{\ell=1}^{N} \alpha_{i,\ell} \;\; = \;\; n + \sum_{j=1}^{i} (m_j - k_j) \qquad \text{for } i = 1, 2, \ldots, r. \tag{5}$$

If we set

$$\begin{aligned} x_{i,\nu} &= \sum_{j=i+1}^{r} (m_j - 1) + \sum_{\ell=1}^{\nu} \alpha_{i,\ell} \\ x_{r,\nu} &= \sum_{\ell=1}^{\nu} \alpha_{r,\ell} \end{aligned}$$

$$i = 0, 1, \ldots, r-1; \quad \nu = 1, 2, \ldots, N$$

$$= \left(\sum_{j=1}^{r}(m_j - 1), \sum_{j=2}^{r}(m_j - 1), \ldots, m_r - 1, 0\right)$$

then relations (3) and (5) become

$$x_{0\nu} \geq x_{1\nu} \geq \ldots \geq x_{r\nu}, \qquad \nu = 0, 1, \ldots, N, \tag{6}$$

with $\mathbf{x}_N = (x_{0,N}, x_{1,N}, \ldots, x_{r,N})$ given by

$$\begin{aligned} x_{i,N} &= \sum_{j=i+1}^{r} (m_j - 1) + \sum_{\ell=1}^{N} \alpha_{i,\ell} \\ &= n - r + i + \sum_{j=1}^{r} m_j - \sum_{j=1}^{i} k_j, \qquad i = 0, 1, \ldots, r \end{aligned}$$

where an empty sum has to be interpreted as zero. It follows that the number of sequences $\mathbf{x}_\nu, \nu = 0, 1, \ldots, N$, satisfying (6) is $N(r+1; \mathbf{x}_0, \mathbf{x}_N)$, according to Theorem 1.

Consider now the superposition of the arrival process at node 1 and the departure processes at nodes $1, 2, \ldots, r$. This process is Poisson with intensity

$$\delta = \lambda + \sum_{i=1}^{r} \mu_i.$$

In $(0, t)$ the process has a total number of jumps equal to

$$N = \sum_{i=0}^{r} \sum_{\ell=1}^{N} \alpha_{i,\ell} = (r+1)n + \sum_{i=1}^{r} (r - i + 1)(m_i - k_i)$$

The (conditional) probability that a jump corresponds to an arrival at node 1 equals λ/δ, and that it corresponds to a departure at node i equals $\mu_i/\delta, i = 1, 2, \ldots, r$. Thus

$$q_t(\mathbf{m}, \mathbf{k}) = e^{-\delta t} \sum_{n \geq 0} \frac{(\delta t)^N}{N!} \left(\frac{\lambda}{\delta}\right)^{\sum \alpha_{0,\ell}} \prod_{i=1}^{r} \left(\frac{\mu_i}{\delta}\right)^{\sum \alpha_{i,\ell}} N(r+1; \mathbf{x}_0, \mathbf{x}_N)$$

$$i=1$$

$$\gamma_r = \lambda \prod_{i=1}^{r} \mu_i,$$

then using Theorem 1 we find after some simplification finally

Theorem 2

$$q_t(\mathbf{m},\mathbf{k}) = e^{-\delta t}\beta_r \sum_{n\geq 0} t^N \gamma_r^n C(n,r;\mathbf{m},\mathbf{k}) \tag{7}$$

where $C(n,r;\mathbf{m},\mathbf{k})$ is an $(r+1)\times(r+1)$ determinant, whose entries are given by

$$c_{ij} = \frac{1}{(n+\sum_{\nu=1}^{j} m_\nu - \sum_{\nu=1}^{i} k_\nu)!}, \qquad i,j = 0,1,\ldots,r, \tag{8}$$

where an empty sum has to be interpreted as zero.

Let us now discuss shortly some algorithmic aspects of (7). If we expand the determinant $C(n,r;\mathbf{m},\mathbf{k})$, then it can be shown, that (7) is proportional to the sum of $(r+1)!$ lattice Bessel functions of rank r. Thus there is no substantial difference in the convergence properties of (7) and the representation of $q_t(\mathbf{m},\mathbf{k})$ given by Massey (1987). However, from a computational point of view, formula (7) is preferable, because the number of floating point operations required to evaluate a determinant of order $(r+1)\times(r+1)$ is $O((r+1)^3)$, if we use a Gaussian procedure. This should be seen in contrast to the formulas given by Massey, which require the evaluation of $(r+1)!$ lattice Bessel functions of rank r. To illustrate these points we have performed a numerical comparison between (7) and formula (1.3) in Massey (1987). To keep the comparison on a fair level the associated permutation group of the lattice Bessel functions have been determined in advance. Computations have been performed on a PC 80386/87 using an APL2 interpreter.

In the following table results are presented for $\lambda = \mu_1 = \ldots = \mu_r = 0.1$, $\mathbf{k} = \mathbf{m} = (1,1,\ldots,1)$:

1	0.74094	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	0.22780	0.475	0.535	0.08211	0.645	5.475	0.03020	0.755	209.355
10	0.05844	0.645	0.705	0.00679	0.805	8.445	0.00091	1.355	233.145

Here τ_1 and τ_2 denote the computing times in seconds using (7) and Massey's formula, respectively.

1 References

[1] Baccelli F., Massey W.A. (1990) A transient analysis of the two-node series Jackson network. *Tech. Report*, INRIA-Sophia, France.

[2] Kreweras G., (1965) Sur une classe de problèmes de dénombrement liés au treillis des partitions de entiers. *Cahiers du Bur. Univ. de Rech. Opér.*,**6**, 5-105.

[3] Massey W.F., (1987) Calculating exit times for series Jackson networks. *J. Appl. Prob.*,**27**, 226-234.