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A Discrete Nodal Domain Theorem for Trees

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Abstract

Let $G$ be a connected graph with $n$ vertices and let $x = (x_1, \ldots, x_n)$ be a real vector. A positive (negative) sign graph of the vector $x$ is a maximal connected subgraph of $G$ on vertices $x_i > 0$ ($x_i < 0$). For an eigenvalue of a generalized Laplacian of a tree: We characterize the maximal number of sign graphs of an eigenvector. We give an $O(n^2)$ time algorithm to find an eigenvector with maximum number of sign graphs and we show that finding an eigenvector with minimum number of sign graphs is an NP-complete problem.

Keywords: discrete nodal domain theorem; eigenvectors of a matrix with non-positive off-diagonal elements; tree; graph Laplacian; Sign graph;
1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$ and let $x = (x_1, \ldots, x_n)$ be a real vector. We associate the real numbers $x_i$ with the vertices $i$ of $G$, for $i = 1, \ldots, n$. A positive (negative) sign graph $S$ is a maximal connected subgraph of $G$ on vertices $i \in V$ with $x_i > 0$ ($x_i < 0$). Sign graphs are also called nodal domains. We denote by $\eta(x)$ the number of sign graphs of the vector $x$.

For example, let $G$ be the path $P_6$ and consider the vector $x = (1, 2, -1, 0, -1, 3)$. The vector $x$ has two positive sign graphs, two negative sign graphs, and hence $\eta(x) = 4$.

Let $G$ be a simple, undirected, loop-free graph with $n$ vertices. We call a symmetric real $n \times n$ matrix $A$ a generalized Laplacian of $G$ if $a_{uv} < 0$ when $u$ and $v$ are adjacent vertices of $G$ and $a_{uv} = 0$ when $u$ and $v$ are distinct and not adjacent. There are no constraints on the diagonal entries of $A$. We say $G$ is the graph of $A$ and we say $A$ is the matrix of $G$.

The number of sign graphs of a graph $G$ is at most the number of vertices of the induced bipartite subgraph of $G$ with maximal number of vertices. To find such an induced bipartite subgraph of $G$ is a well known NP-complete problem (see, e.g., [4]).

On the other hand, if $A$ is a generalized Laplacian of $G$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, then any eigenvector corresponding to eigenvalue $\lambda_k$ with multiplicity $r$ has at most $k + r - 1$ sign graphs of $G$. This theorem is called the discrete nodal domain theorem and it is the discrete analogue of Courant’s nodal domain theorem for elliptic operators on Riemannian manifolds. For a proof of the discrete nodal domain theorem and some historical remarks see Davies et al. [1].

We focus our attention on the $k$-th eigenvalue of generalized Laplacian $A$, and suppose that it has multiplicity $r$, so that

$$\lambda_1 \leq \cdots \leq \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r} \leq \cdots \leq \lambda_n.$$ 

Throughout this paper we assume that the eigenvalues are numbered in non-decreasing order.

Theorem 1 (Discrete nodal domain [1]) Let $G$ be a connected graph and let $A$ be generalized Laplacian of $G$ then any eigenvector corresponding to the eigenvalue $\lambda_k$ with multiplicity $r$ has at most $k + r - 1$ sign graphs.
The theorem is sharp for paths. However, in general it is unknown, whether this upper bound relating to the order of the eigenvalues is sharp for an arbitrary graph. Moreover, no method is known to construct an eigenvector to the eigenvalue $\lambda_k$ with maximal or minimal number of the sign graphs. In this paper we look at the discrete nodal domain theorem for trees. We characterize for a tree: the maximal number of the sign graphs of an eigenvector corresponding to an eigenvalue $\lambda_k$. We give an $O(n^2)$ time algorithm to find an eigenvector with maximum number of the sign graphs, which corresponds to an eigenvalue $\lambda_k$. We show that to find an eigenvector of an eigenvalue $\lambda_k$, which has minimum number of the sign graphs, is NP-complete.

2 Nodal domain theorem for trees

In this paper we look at the discrete nodal domain theorem for trees. We begin with a special simple eigenvalue.

We say that $y$ is a $\lambda$-eigenvector (of $A$) if $Ay = \lambda y$.

**Theorem 2** Let $G$ be a tree and let $A$ be a generalized Laplacian of $G$. If $y$ is a $\lambda_k$-eigenvector without a vanishing coordinate, then $\lambda_k$ is simple and $y$ has exactly $k$ sign graphs.

The following lemma plays an important role in the proof of the theorem 2.

**Lemma 1 (Fiedler [3])** Let $A$ be a generalized Laplacian of a tree. If $y$ is a $\lambda_k$-eigenvector without a vanishing coordinate, then $\lambda_k$ is simple and there are exactly $n - k$ (unordered) pairs $(i, j)$, $i \neq j$, for which $a_{ij}y_i y_j < 0$.

**Proof of Theorem 2:** By lemma 1, $\lambda_k$ is simple and there are exactly $n - k$ edges $ij$, for which $y_i$ and $y_j$ have the same sign. Note that $a_{ij}y_i y_j < 0$ if and only if $i$ and $j$ are adjacent and $y_i$ and $y_j$ have the same sign. We divide $V$ in three disjoint sets in the following way:

$P = \{ i \in V : y_i > 0, \text{ and there is an edge } ij \in E, \text{ s.t. } y_j > 0 \}$,

$M = \{ i \in V : y_i < 0, \text{ and there is an edge } ij \in E, \text{ s.t. } y_j < 0 \}$.

$C$ is the set of remaining vertices. The induced subgraphs $G[P]$ and $G[M]$ are forests. Let $p$ and $m$ are the number of components of $G[P]$ and $G[M]$, respectively. $G[P]$ and $G[M]$ have $|P| - p$ edges and $|M| - m$ edges, respectively. Since $\{P, M, C\}$ is a partition of $V$ and using lemma 1, we see $|P| - p + |M| - m = n - k$.  

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Now we show that $\eta(y) = k$. Let $i$ and $j$ be vertices of $C$. If $y_i$ and $y_j$ have the same sign, then $i$ and $j$ are not adjacent. Let $C_- = \{ i \in C : y_i < 0 \}$ and $C_+ = \{ i \in C : y_i > 0 \}$. By the definition of $P$ and $M$, there exist no edges between $C_-$ and $M$ and no edges between $C_+$ and $P$, respectively. Consequently the number of sign graphs of $y$ is equal to $|C| + p + m$. Thus
\[
\eta(y) = |C| + p + m = n - |P| - |M| + |P| + |M| - n + k = k.
\]

We remark that R. Roth [5] proved that the largest eigenvalue of the generalized Laplacian of a bipartite graph satisfies the condition of theorem 2 and largest eigenvalue has an eigenvector with $n$ sign graphs.

Next we consider eigenvectors of trees with vanishing coordinates. Let $G = (V, E)$ be a connected graph, and let $A$ be a generalized Laplacian of $G$. Let $Z$ be a subset of $V$, let $G_1, \ldots, G_m$ be the components of $G - Z$ and let $A_1, \ldots, A_m$ be generalized Laplacians of $G_1, \ldots, G_m$. We say $(A_1, \ldots, A_m, A_Z)$ is a rearrangement of $A$, if we rearrange the matrix $A$ with permutation similarity operations in the following way:

\[
A = \begin{pmatrix}
A_1 & A_{12} & \cdots & A_{1Z} \\
\vdots & \ddots & \cdots & \vdots \\
A_{m1} & \cdots & A_m & A_{mZ} \\
A_{Z1} & \cdots & A_{Zm} & A_Z
\end{pmatrix}
\]

**Theorem 3** Let $G$ be a tree with $n$ vertices and let $A$ be a generalized Laplacian of $G$. Let $\lambda$ be an eigenvalue of $A$ with multiplicity $r \geq 2$. Then there exists a rearrangement $(A_1, \ldots, A_m, A_Z)$ of $A$ such that the following statements hold:

(i) $\lambda$ is a simple eigenvalue of $A_1, \ldots, A_m$.

The matrix $A_j$ has a $\lambda$-eigenvector without vanishing coordinates, for $j = 1, \ldots, m$.

(ii) Let $k_1, \ldots, k_m$ be the positions of $\lambda$ in the spectra of $A_1, \ldots, A_m$ in non-decreasing order. Then the number of sign graphs of an eigenvector of $\lambda$ is at most $k_1 + \cdots + k_m$.

(iii) There exists an eigenvector of $\lambda$ with $k_1 + \cdots + k_m$ sign graphs. Such an eigenvector can be found in $O(n^2)$ time.
For the proof of Theorem 3 we need the following two lemmas. We shall prove lemma 3 after the proof of Theorem 3.

**Lemma 2 (Fiedler [3])** Each eigenvector corresponding to a multiple eigenvalue of a matrix of a tree has at least one vanishing coordinate.

We remark that M. Fiedler proved the results of lemmas 1 and 2 for a more general matrix of a tree.

**Lemma 3** Let $x_1, \ldots, x_k$ be linearly independent vectors in $\mathbb{R}^n$ and $k < n$. If all linear combinations of $x_1, \ldots, x_k$ have a vanishing coordinate, then the vectors $x_1, \ldots, x_k$ have a common vanishing coordinate.

**Proof of Theorem 3:** Let $\lambda$ be an eigenvalue of $A$ with multiplicity $r \geq 2$. Let $y_1, \ldots, y_r$ be linearly independent $\lambda$-eigenvectors. Let $Z$ be the set of all common vanishing coordinates of $y_1, \ldots, y_r$. By lemmas 2 and 3, $Z$ is not empty and the choice of $y_1, \ldots, y_r$ has no influence on $Z$. The graph $G - Z$ is a forest with components $T_1, \ldots, T_m$. Let $A_1, \ldots, A_m$ be generalized Laplacians of $T_1, \ldots, T_m$. According to the rearrangement $(A_1, \ldots, A_m, A_Z)$ the matrix $A$ has the following form:

$$
A = 
\begin{pmatrix}
A_1 & 0 & \cdots & 0 & A_{1Z} \\
0 & A_2 & \cdots & 0 & A_{2Z} \\
0 & \cdots & \ddots & 0 & \vdots \\
0 & \cdots & 0 & A_m & A_{mZ} \\
A_{Z1} & \cdots & \cdots & A_{Zm} & A_Z
\end{pmatrix}
$$

(i) We write each eigenvector $y$ of $\lambda$ as $y = (y_{T_1}, \ldots, y_{T_m}, 0, \ldots, 0)$, where $y_{T_j}$ denotes the coordinates of eigenvector $y$ belonging to the tree $T_j$. By the definition of $Z$, the coordinates of eigenvector $y$ belonging to $Z$ are equal to zero. Thus the vector $Ay$ has the following form:

$$
Ay = (A_1 y_{T_1}, \ldots, A_m y_{T_m}, *, \ldots, *) = (\lambda y_{T_1}, \ldots, \lambda y_{T_m}, 0, \ldots, 0) = \lambda y
$$

for each $\lambda$-eigenvector $y$. Therefore $\lambda$ is an eigenvalue of the matrices $A_1, \ldots, A_m$. Now we prove that $\lambda$ is a simple eigenvalue of $A_j$ and the matrix $A_j$ has a $\lambda$-eigenvector without vanishing coordinates, for $j = 1, \ldots, m$. We show that the number of linearly independent vectors of $y_{T_j}, \ldots, y_{T_j}^{*}$ is equal to one, for $j = 1, \ldots, m$. Note that $y_{T_j}, \ldots, y_{T_j}^{*}$ are the restrictions of the eigenvectors $y_1, \ldots, y_r$ to the subtree $T_j$. Assume that there are linearly
independent vectors $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$, $h \geq 2$. Then the vectors $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$ are linearly independent $\lambda$-eigenvectors of $A_j$. By lemmas 2 and 3 the vectors $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$ have a common vanishing coordinate. Hence $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$ have a common vanishing coordinate, a contradiction to the definition of $Z$.

We denote by $b_j$ the only one linearly independent vector of $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$, for $j = 1, \ldots, m$. The vector $b_j$ is a $\lambda$-eigenvector of $A_j$, for $j = 1, \ldots, m$. The eigenvector $b_j$ has no vanishing coordinate, for $j = 1, \ldots, m$. We suppose that $b_j$ has a vanishing coordinate. Then $y^{1}_{T_j}, \ldots, y^{h}_{T_j}$ have a common vanishing coordinate, a contradiction to the definition of $Z$.

(ii) Let $k_1, \ldots, k_m$ be the positions of $\lambda$ in the spectrum of $A_1, \ldots, A_m$ in non-decreasing order. The number of sign components of an eigenvector $y = (\beta_1 b_1, \ldots, \beta_m b_m, 0, \ldots, 0)$ is equal to the sum of the number of sign components of $\beta_1 b_1, \ldots, \beta_m b_m$. By theorem 2, $\eta(b_j) = k_j$, for $j = 1, \ldots, m$.

Therefore, $\eta(y) \leq k_1 + \cdots + k_m$.

(iii) Now we construct an eigenvector $x$ of $\lambda$ with $\eta(x) = k_1 + \cdots + k_m$ in following way: By the definition of $b_j$, the linearly independent eigenvectors $y^{1}, \ldots, y^{r}$ are of the form $y^{i} = (\beta_{i1} b_1, \ldots, \beta_{im} b_m, 0, \ldots, 0)$, for $i = 1, \ldots, r$, where the coefficients $\beta_{i1}, \ldots, \beta_{im}$ are real numbers.

$$x := y^{1};$$

for $i = 2, \ldots, r$ do

$$x := x + \alpha_i y^{i};$$

choose $\alpha_i$: $\alpha_i \neq 0$ and $\alpha_i \notin \{-\frac{x_j}{y^{j}_j} : y^{j}_j \neq 0, j = 1, \ldots, n\}$.

After this iteration we obtain $x = (\beta'_{11} b_1, \ldots, \beta'_{m1} b_m, 0, \ldots, 0)$. The coefficients $\beta'_{11}, \ldots, \beta'_{m1}$ are nonzero numbers. Assume that there exists a $\beta'_{jj} = 0$. By the choice of $\alpha_i$, all $\beta_{ij}, \ldots, \beta_{tj}$ are equal to zero. This is a contradiction to the definition of $Z$.

Therefore, $\eta(x) = \eta(\beta'_{11} b_1) + \cdots + \eta(\beta'_{m1} b_m) = k_1 + \cdots + k_m$.

It is easy to see that we need $O(n^2)$ operations to find an eigenvector $x$ with $\eta(x) = k_1 + \cdots + k_m$ from an arbitrary eigensystem of $A$.

Finally, we complete the eigenvalues of a tree.

**Corollary 1** By theorem 3, if we replace the multiple eigenvalue $\lambda$ by the simple eigenvalue $\lambda$ with an eigenvector $y$, which has at least one vanishing coordinate, then the statements of theorem 3 also hold.
Proof of Lemma 3: Let $x_1, \ldots, x_k$ be linearly independent vectors in $\mathbb{R}^n$, $k < n$ such that all linear combinations of $x_1, \ldots, x_k$ have a vanishing coordinate. We prove that the vectors $x_1, \ldots, x_k$ have a common vanishing coordinate.

If $k = 1$, this is trivial. Let $k \geq 2$. Let $y$ be a linear combination of $x_1, \ldots, x_{k-1}$. Let $Z_y = \{j : y_j = x_j = 0\}$. Without loss of generality let the first $d$ coordinates of $x^k$ be zero and all others elements of $x^k$ be nonzero.

Claim 1: $y$ and $x^k$ have a common vanishing coordinate, i.e. $Z_y$ is not empty. Suppose that $y$ and $x^k$ have no common vanishing coordinate. Then the first $d$ elements of $y$ are nonzero. Now we construct a new vector $t = y + \beta x^k$. We choose $\beta$ in the following way: $\beta \neq 0$ and $\beta \neq \frac{y_i}{x_i}$ for $i = d + 1, \ldots, n$. Then $t$ has no vanishing coordinate. This is a contradiction.

Claim 2: If $u$ and $y$ are linear combinations of $x_1, \ldots, x_{k-1}$, then $Z_u \cap Z_y = \emptyset$. Suppose that there exists $u$ and $y$, such that $Z_u \cap Z_y = \emptyset$. By claim 1, $Z_u$ and $Z_y$ are not empty. Without loss of generality, the first $d$ elements of $u$ and $y$ look like: $u = (0, \ldots, 0, \pm, \ldots, \pm)$, $y = (\pm, \ldots, \pm, 0, \ldots, 0, \pm, \ldots, \pm)$. Now we construct a new vector $t = u + \beta y$. We choose $\beta$ such that: $\beta \neq 0$ and $\beta \neq \frac{u_i}{y_i}$ for $i = 1, \ldots, d$ and $y_i$ are nonzero. Then $t$ and $x^k$ have no common zero coordinate. This is a contradiction to claim 1.

Now we define new vectors $y^i$ in the following way:

$y^1 = x^1$, $y^i = y^{i-1} + \alpha_i x^i$, for $i = 2, \ldots, k - 1$. We choose $\alpha_i$ such that:

$\alpha_i \neq 0$ and $\alpha_i \neq -\frac{y^i_{j-1}}{x^i_j}$, for all $x^i_j$ nonzero elements, for $j = 1, \ldots, d$.

Claim 3: $Z_{y^i}$ is not empty and $Z_{y^i} = Z_{x^1} \cap \cdots \cap Z_{x^i}$, for $i = 1, \ldots, k - 1$. By claim 1, $Z_{y^i}$ is not empty. We prove the other argument with induction on $i$. For $i = 1$, $y^1 = x^1$. By claim 1, $x^1$ and $x^k$ have a common zero coordinate. We suppose that the claim holds for $y^1, \ldots, y^{i-1}$. Now we show that it holds for $y^i = y^{i-1} + \alpha_i x^i$. We choose $\alpha_i$ as defined. By Claim 2, $Z_{y^{i-1}} \cap Z_{x^i} \neq \emptyset$. By the choice of $\alpha_i$, $y^i_j = 0$ if and only if $j \in Z_{y^{i-1}}$ and $j \in Z_{x^i}$. It means that $j \in Z_{y^{i-1}} \cap Z_{x^i}$. By induction $Z_{y^{i-1}} = Z_{x^1} \cap \cdots \cap Z_{x^{i-1}}$. Then $j \in Z_{x^1} \cap \cdots \cap Z_{x^{i-1}} \cap Z_{x^i}$.

By claim 3, $Z_{y^{k-1}}$ is not empty and $Z_{y^{k-1}} = Z_{x^1} \cap \cdots \cap Z_{x^{k-1}}$. Therefore $x^1, \ldots, x^k$ have a common vanishing coordinate. □
3 Minimum number of sign graphs

In this section we show that the following problem is NP-complete.

**MINIMUM NUMBER OF SIGN GRAPHS**

**Instance:** An $n \times n$ matrix $A$, where $A$ is a generalized Laplacian of a tree, an eigenvalue $\lambda$ of $A$ with multiplicity $r \geq 2$.

**Question:** Find an eigenvector $y$ of $\lambda$ such that the number of sign graphs of $y$ is minimal.

Let $A$ be a generalized Laplacian of a tree and $\lambda$ is an eigenvalue of $A$ with multiplicity $r \geq 2$. In theorem 3 we proved that linearly independent eigenvectors $y_1, \ldots, y_r$ of $\lambda$ have common vanishing coordinates $Z$ and $y_i = (\beta_{i1}b_1, \ldots, \beta_{im}b_m, 0, \ldots, 0)$, for $i = 1, \ldots, r$, where $b_1, \ldots, b_m$ are vectors without vanishing coordinates and $\beta_{i1}, \ldots, \beta_{im}$ are real numbers. $m$ is the number of components of $G - Z$.

Let $B = (\beta_{ij})$, $i = 1, \ldots, m$, $j = 1, \ldots, r$. Then an eigenvector $y$ of $\lambda$ has the following form: $y = ((Bx)_1b_1, \ldots, (Bx)_mb_m, 0, \ldots, 0)$, where $x = (x_1, \ldots, x_r)$ is a real vector. Let $k_1, \ldots, k_m$ are the number of sign components of $b_1, \ldots, b_m$. Now we define new variables $c_i(x)$, $i = 1, \ldots, m$ as follows:

$$c_i(x) = \begin{cases} 0, & \text{if } (Bx)_i = 0, \\ 1, & \text{if } (Bx)_i \neq 0. \end{cases}$$

Then $\eta(y) = k_1c_1(x) + \cdots + k_mc_m(x)$. Therefore MINIMUM NUMBER OF SIGN GRAPHS is equivalent to the following minimization problem:

$$\min k_1c_1(x) + \cdots + k_mc_m(x)$$

$x = (x_1, \ldots, x_r)$ is a nonzero real vector.

Consequently the decision problem of MINIMUM NUMBER OF SIGN GRAPHS is the following problem:

**MIN($\eta$)**

**Instance:** An $(m \times r)$ matrix $B$ with real entries, positive integers $k_1, \ldots, k_m$ and a positive integer $s$.

**Question:** Is there a nonzero rational vector $x = (x_1, \ldots, x_r)$, such that $k_1c_1(x) + \cdots + k_mc_m(x) \leq s$?

**Lemma 4** The $(m \times r)$ matrix $B$ of decision problem $\text{MIN}(\eta)$ can be arbitrary large.
Proof: The required example is constructed from the following result by I. Faria [2]. Let \( G \) be a graph and let the matrix \( L = D - A \) be the Laplacian matrix of \( G \), where \( A \) is the adjacency matrix of \( G \) and \( D \) is the diagonal matrix of vertex degrees of \( G \). Let \( p \) be the number of vertices with degree one. Let \( q \) be the number of vertices, which are adjacent to a vertex with degree one. Then \( \lambda = 1 \) is an eigenvalue of \( L \) with multiplicity \( r \geq p \cdot q \).

We consider a binary tree with \( n \) vertices and \( n/2 \) endvertices. Therefore \( \lambda = 1 \) is an eigenvalue of \( L \) with multiplicity \( r \geq n/4 \). It is straightforward to show that \( m \) is at least the number of endvertices. Thus \( m \geq n/2 \).

Now we show that MIN(\( \eta \)) is NP-complete. For the proof we give another NP-complete problem. Let \( x = (x_1, \ldots, x_n) \) be a real vector. We denote by \( \text{support}(x) \), the number of nonzero elements of \( x \).

MINIMUM SUPPORT

Instance: An \((m \times r)\) matrix \( B \) with rational entries, a positive integer \( s \).

Question: Is there a nonzero rational vector \( x = (x_1, \ldots, x_r) \) such that \( \text{support}(Bx) \leq s \) ?

Lemma 5 MINIMUM SUPPORT is NP-complete.

Theorem 4 The decision problem MIN(\( \eta \)) is NP-complete.

Proof: It is easy to see that MIN(\( \eta \)) is in NP. We reduce MINIMUM SUPPORT to MIN(\( \eta \)) in following way. We choose \( k_1 = \cdots = k_m = 1 \). The matrix \( B \) is the same matrix. We have the bound \( s \). We assume that there is a vector \( x \) such that \( c_1(x) + \cdots + c_m(x) \leq s \). By the definition of \( c_1(x), \ldots, c_m(x) \), the inequality \( c_1(x) + \cdots + c_m(x) \leq s \) holds if and only if \( \text{support}(Bx) \leq s \). Therefore we have the solution of MINIMUM SUPPORT. Thus MIN(\( \eta \)) is NP-complete.

Proof of Lemma 5: It is easy to see that MINIMUM SUPPORT is in NP. The following problem is NP-complete:

ONE-IN-THREE

Instance: Set \( X \) with \( n \) elements and a subset \( T \) of \( X \times X \times X \).

Question: Is there a subset \( Y \) of \( X \), such that each triple \( t = (t_1, t_2, t_3) \) in \( T \) has exactly one element in \( Y \)?


We reduce ONE-IN-THREE to MINIMUM SUPPORT in following way. For
each element of $X$ we give a variable $x_i$, for $i = 1, \ldots, n$. We add a new variable $x_{n+1}$. We introduce rows $x_i + x_{n+1}$ and $x_i - x_{n+1}$ in the matrix $B$, for $i = 1, \ldots, n$. For each triple $t = (t_i, t_j, t_k)$ in $T$ we introduce the row $x_i + x_j + x_k + x_{n+1}$, $n + 1$ times in $B$. We set the bound $s = n$. We assume that $\text{support}(Bx) \leq n$. Then each variable $x_i$ is equal to $x_{n+1}$ or $-x_{n+1}$, for $i = 1, \ldots, n$ and each expression $x_i + x_j + x_k + x_{n+1}$ is equal to zero. Otherwise $\text{support}(Bx) > n$. Now we put the variables $x_i = x_{n+1}$ in $Y$. It is easy to see that each triple $t = (t_1, t_2, t_3)$ in $T$ has exactly one element in $Y$ if and only if $x_i + x_j + x_k + x_{n+1}$ is equal to zero. Therefore we have the solution of ONE-IN-THREE. Thus MINIMUM SUPPORT is NP-complete.

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References


