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THE MAXIMAL HEIGHT OF SIMPLE RANDOM WALKS REVISITED

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In a recent paper Katzenbeisser and Panny (1996) derived distributional results for a number of so called simple random walk statistics defined on a simple random walk in the sense of Cox and Miller (1968) starting at zero and leading to state l after n steps, where l is arbitrary, but fix. In the present paper the random walk statistics D_n^+ = the one-sided maximum deviation and Q_n = the number of times where the maximum is achieved, are considered and distributional results are presented, when it is irrespective, where the random walk terminates after n steps. Thus, the results can be seen as generalizations of some well known results about (purely) binomial random walk, given e.g. in Revesz (1990).

1. Introduction. Let $\mathbf{X}_j$, $j = 1, 2, \ldots$, be independent and identically distributed random variables with

$$\mathbf{P}(\mathbf{X}_j = 1) = \alpha, \quad \mathbf{P}(\mathbf{X}_j = 0) = \beta, \quad \mathbf{P}(\mathbf{X}_j = -1) = \gamma,$$

where $\alpha + \beta + \gamma = 1$. Consider the random walk

$$\mathbf{S}_k = \mathbf{S}_0 + \sum_{j=1}^{k} \mathbf{X}_j, \quad k = 1, 2, \ldots, n \quad \text{with} \quad \mathbf{S}_n = \ell\,,$$

i.e. a simple random walk in the sense of Cox and Miller (1965) starting at $\mathbf{S}_0$ and leading to ℓ after n steps. Confining to $\mathbf{S}_0 = 0$ actually constitutes no restriction at all. So this assumption will be made in the sequel.

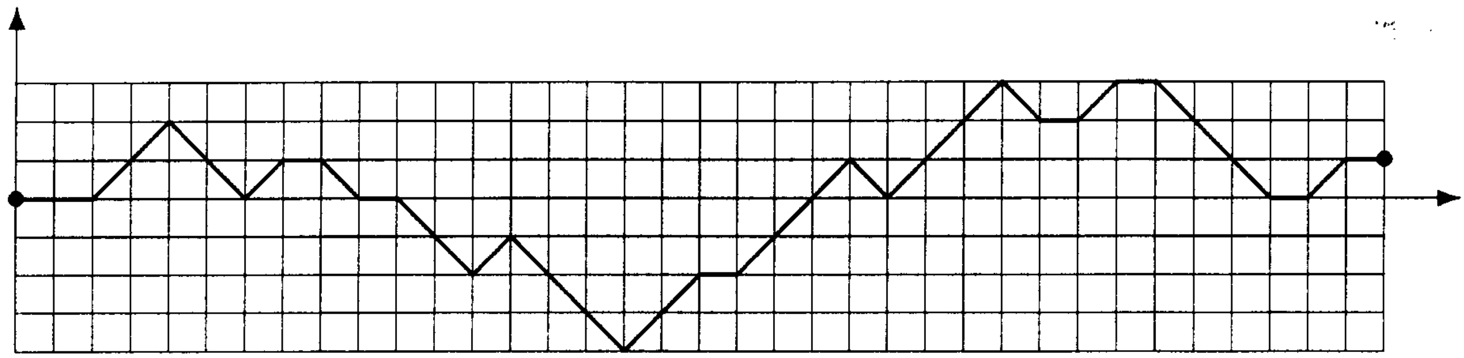
A number of random variables, so called *simple random walk statistics* can be defined on this random walk; for a list of simple random walk statistics see e.g. Katzenbeisser and Panny (19969, p.314f). Prominent random variables defined on such a simple random walk are associated with the maximal one sided height achieved by it:

$$\mathbf{D}_n^+ = \textit{the one-sided maximal height},$$

where $\mathbf{D}_n^+ = \max_{0 \le k \le n} \mathbf{S}_k$, and

$$\mathbf{Q}_n, = \textit{the number of times where the maximum is achieved.}$$

The maximum is achieved, if $\mathbf{S}_k = \mathbf{S}_{k+1} = \mathbf{S}_{k+2} = \ldots = \mathbf{S}_{k+m} = \mathbf{D}_n^+$ and $\mathbf{S}_{k-1}, \mathbf{S}_{k+m+1} < \mathbf{D}_n^+$, $0 \le k \le k+m \le n$. By definition, if $\mathbf{S}_0 = \mathbf{D}_n^+$ the random walk starts with a maximum; accordingly, if $\mathbf{S}_n = \mathbf{D}_n^+$ the random walk ends with a maximum. If there should be one or more consecutive horizontal steps coinciding with the line $y = \mathbf{D}_n^+$ (i.e. $m > 0$), this counts only as a single maximum. The following figure 1 shows a possible sample path for a simple random walk with $\mathbf{D}_n^+ = 3$, $\mathbf{Q}_n = 2$, and $\mathbf{S}_n = 1$.



Results concerning some joint- and conditional distributions of the random variables $\mathbf{D}_n^+$ and $\mathbf{Q}_n$ are well known. The joint distribution $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r, \mathbf{S}_n = \ell)$ as well as the two-dimensional marginal distributions $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{S}_n = \ell)$ and $\mathbf{P}(\mathbf{Q}_n = r, \mathbf{S}_n = \ell)$, ℓ *arbitrary* but fixed, can be found in Katzenbeisser and Panny (1996) and Panny and Katzenbeisser (1997). Also, the conditional distributions $\mathbf{P}(\mathbf{D}_n^+ = k|\mathbf{S}_n = \ell)$ and $\mathbf{P}(\mathbf{Q}_n = k|\mathbf{S}_n = \ell)$ can be obtained using results from Katzenbeisser and Panny (1996); especially for $\mathbf{S}_n = 0$, distributional properties of $\mathbf{D}_n^+|\mathbf{S}_n = 0$ are discussed in Katzenbeisser and Panny (1984, 1986). Specializations for purely binomial random walks, i.e. random walks with $\alpha = \gamma = \frac{1}{2}$, $\beta = 0$, and $\mathbf{S}_n = 0$ are given in Dwass (1967) and Katzenbeisser and Panny (1992, 1993).

The purpose of the present paper is to study the joint distribution $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$, $k \geq 0$, $r > 0$, the marginal distributions $\mathbf{P}(\mathbf{D}_n^+ = k)$ and $\mathbf{P}(\mathbf{Q}_n = r)$, and the corresponding moments of $\mathbf{D}_n^+$ and $\mathbf{Q}_n$, irrespective where the simple random walk terminates after n steps. Thus the results can be seen as generalizations of some well known results given e.g. in Revesz (1990) for purely binomial random walks, ie.e random walks which moves one unit up and to the right or one unit down and to the right with probability 1/2.

2. Technical prerequisites. In a recent paper, Panny and Katzenbeisser (1997) derived the joint distribution $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r, \mathbf{S}_n = \ell)$, which is given by

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r, \mathbf{S}_n = \ell) = \\ = \rho^{k+r-1} \sum_{j \geq 0} \rho^j \binom{-r}{j} \left[\binom{n; \alpha, \beta, \gamma}{n + \ell - 2k - 2r - 2j + 2} - \rho \binom{n; \alpha, \beta, \gamma}{n + \ell - 2k - 2r - 2j} \right], \quad (1)$$

where $\rho = \alpha/\gamma$ and so called generalized trinomial coefficients (GTC's) are used. They have generating function $(\alpha v^2 + \beta v + \gamma)^n$, i.e.

$$\binom{n; \alpha, \beta, \gamma}{k} = [v^k](\alpha v^2 + \beta v + \gamma)^n,$$

and $[v^k]P(v)$ denotes the coefficient of v^k in $P(v)$. Properties of the GTC's can be found in Panny (1984), Katzenbeisser and Panny (1984, 1996), and Böhm (1993); a probabilistic interpretation is given by

$$\mathbf{P}(\mathbf{S}_n = \ell) = \binom{n;\alpha,\beta,\gamma}{n+\ell}$$

for all admissible ℓ, i.e. the probability that an unrestricted simple random walk reaches the state ℓ after n steps and $\ell \in \{-n, -n+1, \ldots, n\}$.

The result (1) was derived from a generating function (g.f.) $\phi(k, r, \ell; y)$ of $\mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r, \mathbf{S}_n = \ell)$,

$$\phi(k, r, \ell; y) = \sum_{n\geq 0} \mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r, \mathbf{S}_n = \ell) y^n,$$

which can be obtained by convoluting appropriate versions of a single basic g.f. $\Psi_{h,m,\ell}(y)$ introduced by Panny (1984), and defined by

$$\Psi_{h,m,\ell}(y) = \sum_{n\geq 0} p(h, m, \ell, n) y^n, \quad h, m \geq 0,$$

where

$$p(h, m, \ell, n) = \mathbf{P}(-m \leq \mathbf{S}_1 \leq h, \ldots, -m \leq \mathbf{S}_{n-1} \leq h, -m \leq \mathbf{S}_n = \ell \leq h \,|\, \mathbf{S}_0 = 0),$$

i.e. the probability that a particle obeying a simple random walk with (weak sense) absorbing barriers at $-m$ and h reaches the state ℓ after n steps when it started from state 0:

$$\Psi_{h,m,\ell}(y) = \frac{\rho^{\frac{|\ell|+\ell}{2}}}{\gamma} v^{|\ell|} \frac{\alpha v^2 + \beta v + \gamma}{1 - \rho v^2} \frac{\left[1-(\rho v^2)^{m+1-\frac{|\ell|-\ell}{2}}\right]\left[1-(\rho v^2)^{h+1-\frac{|\ell|+\ell}{2}}\right]}{1-(\rho v^2)^{h+m+2}},$$

where again $\rho = \alpha/\gamma$ and the substitution $y = g(v) = v/(\alpha v^2 + \beta v + \gamma)$ has been used. This substitution is crucial because it considerably simplifies the original generating function in terms of y; for further technical details cf. Katzenbeisser and Panny (1984). Once the interesting generating function has been obtained one only has to extract the coefficient of y^n in the g.f. Explicit expressions can be derived by an application of Cauchy's integral formula, e.g for $p(h, m, \ell, n)$, we get

$$p(h, m, \ell, n) = \frac{1}{2\pi i} \oint \frac{\Psi_{h,m,\ell}(y)}{y^{n+1}} dy\,, \tag{2}$$

where we only have to take the substitution $y = g(v) = v/(\alpha v^2 + \beta v + \gamma)$ into account. Since $y = v/\gamma + O(v^2)$ as $v \to 0$, and $g(v)$ is analytic in a sufficiently small neighborhood of 0, (2) remains valid also after substituting $y = g(v)$. Hence

$$p(h, m, \ell, n) = \frac{1}{2\pi i} \oint \frac{\Psi_{h,m,\ell}(g(v))}{g^{n+1}(v)} g'(v) dv\,,$$

where

$$\frac{g'(v)}{g^{n+1}(v)} = \gamma \frac{1-\rho v^2}{v^{n+1}} (\alpha v^2 + \beta v + \gamma)^{n-1}.$$

This approach was successfully applied in Katzenbeisser and Panny (1996), where distributional properties for a number of simple random walk statistics have been derived. Moreover, this approach will also prove useful in obtaining distributional results concerning the random variables $\mathbf{D}_n^+$ and $\mathbf{Q}_n$ considered in this paper. All results given in the sequel are based on appropriately defined versions of the g.f. $\phi(k,r,\ell;y)$ and an application of Cauchy's integral formula.

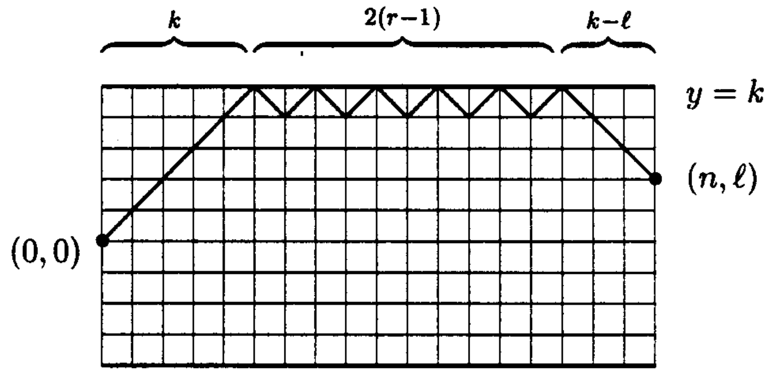
3. Probability distributions. Starting point in deriving the joint- and the marginal distributions of $\mathbf{D}_n^+$ and $\mathbf{Q}_n$ is the g.f. $\phi(k,r,\ell;y)$ which is given by

$$\phi(k,r,\ell;y) = \frac{1}{\gamma}\Big(\frac{\rho v^2}{1+\rho v^2}\Big)^r (\alpha v^2 + \beta v + \gamma)\big(\rho v^2\big)^{k-1}\frac{1}{v^\ell}, \tag{3}$$

cf. Katzenbeisser and Panny (1996, p.328), and $y = g(v) = v/(\alpha v^2 + \beta v + \gamma)$. The joint probability $\mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r, \mathbf{S}_n = \ell)$ can now be derived by an application of Cauchy's integral formula, where we have to take the substitution into account; technically, this means that

$$\mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r, \mathbf{S}_n = \ell) = [v^n]\left\{\rho^{r+k-1} v^{2r+2k-\ell-2}\frac{1-\rho v^2}{(1+\rho v^2)^r}(\alpha v^2 + \beta v + \gamma)^n\right\}.$$

For n,k,ℓ,r we have the following obvious relations: (a) $n \geq 0$, (b) $1 \leq r$, (c) $0 \leq k$, and (d) $k \geq \ell$. Furthermore we have (e) $k + 2(r-1) + k - \ell \leq n$ which is clear from the following consideration, and can be see from figure 2: to achieve height k for the first time we have to move at least k steps upward and to achieve height k additionally $r-1$ times there are further at lest $2(r-1)$ steps required. Finally, to reach height ℓ from the r-th visit of k there are additionally $k-\ell$ steps necessary.



Moreover, from the relations above, we immediately get: from (c),(d),and (e) we find (b') $r \leq \frac{n}{2} + 1$. From (b), (d), and (e) we have (c') $k \leq n$, and finally, from (b),(c), and (e) we get (d') $\ell \geq -n$. Moreover, if conditions (a)-(d) are fulfilled, than (e) is automatically guaranteed: if (e) is satisfied $\phi(k,r,\ell;y)$ yields the corresponding probabilities. However, if condition (e) is not satisfied, the corresponding coefficients in the generating function are all zero: the lowest term in $\phi(k,r,\ell;y)$ is $\rho^{r+k-1}\gamma^n v^{2r+2k-\ell-2}$, and if (e) is not satisfied, then $k+2(r-1)+k-\ell > n$ which implies that the corresponding coefficients are zero which also can immediately be seen from formula (1). This observation is crucial for our approach, because this property of the g.f. considerably simplifies the summation procedures which are necessary to obtain the interesting marginal distributions as well as the moments. Of course, the results could also be obtained by appropriately summation in formula (1). However, we prefer to sum within the corresponding g.f. The advantage of this approach is that the resulting formulae can easier be handled and the corresponding moment generating functions and therefore the moments of the interesting random variables can immediately be derived.

The joint distribution $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$. Summation in formula (3) over all admissible values of ℓ, i.e. $-n+2k+2(r-1) \le \ell \le k$, which can be extended to $\ell \le k$, and which is justified by the considerations above, yields the g.f. $\phi(k,r;y)$ for $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$:

$$\sum_\ell \phi(k,r,\ell;y) = \sum_\ell \sum_{n\ge 0} \mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r, \mathbf{S}_n = \ell) y^n = $$
$$= \sum_{n\ge 0} \mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) y^n = \phi(k,r;y)\,.$$

Because $\sum_{\ell\le k} \frac{1}{v^\ell} = \frac{1}{v^k}\frac{1}{1-v}$, for $|v|<1$, we have

$$\phi(k,r;y) = \frac{1}{\gamma}(\alpha v^2+\beta v+\gamma)\frac{1}{\rho v^2}\left(\frac{\rho v^2}{1+\rho v^2}\right)^r (\rho v)^k \frac{1}{1-v}, \tag{4}$$

which is basic for all further considerations, and

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) = [y^n]\phi(k,r;y).$$

Picking out the coefficient of y^n by means of Cauchy's integral formula (where we have to take into account the substitution) yields the result. Technically, this means that

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) = [v^n]\left\{\rho^{r+k-1} v^{2r+k-2}(\alpha v^2+\beta v+\gamma)^n \frac{1-\rho v^2}{(1+\rho v^2)^r}\frac{1}{1-v}\right\}.$$

Expanding in powers of v we get the following expression for $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$:

$$\begin{aligned}\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) &= \\ &= \rho^{r+k-1}\sum_{i\ge 0}\rho^i\binom{-r}{i}\sum_{j\ge 0}\left[\binom{n;\alpha,\beta,\gamma}{A+2-2i-j} - \rho\binom{n;\alpha,\beta,\gamma}{A-2i-j}\right] = \\ &= \rho^{r+k-1}\sum_{i\ge 0}\rho^i\binom{-r}{i}\left[\binom{n;\alpha,\beta,\gamma}{A+2-2i} + \binom{n;\alpha,\beta,\gamma}{A+1-2i} + (1-\rho)\sum_{j\ge 0}\binom{n;\alpha,\beta,\gamma}{A-2i-j}\right],\end{aligned} \tag{5}$$

where $A = n - 2r - k$. For the simple special cases, $k = n$, $r = 1$, and $k = 0$, $r = n/2 + 1$, respectively, we get

$$\mathbf{P}(\mathbf{D}_n^+ = n, \mathbf{Q}_n = 1) = \rho^n \binom{n; \alpha, \beta, \gamma}{0} = \alpha^n ,$$

$$\mathbf{P}(\mathbf{D}_n^+ = 0, \mathbf{Q}_n = \frac{n}{2} + 1) = \rho^{\frac{n}{2}} \binom{n; \alpha, \beta, \gamma}{0} = (\alpha\gamma)^{n/2} .$$

The marginal distributions $\mathbf{P}(\mathbf{D}_n^+ = k)$ and $\mathbf{P}(\mathbf{Q}_n = r)$. Summation over all admissible values of r, $1 \le r \le \frac{n-k}{2} + 1$, which again can be extended to $r \ge 1$, for fixed k in (4) yields the g.f. $\phi(k; y)$ for $\mathbf{P}(\mathbf{D}_n^+ = k)$:

$$\begin{aligned}\sum_r \phi(k, r; y) &= \sum_r \sum_{n\ge 0} \mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r) y^n = \\ &= \sum_{n \ge 0} \mathbf{P}(\mathbf{D}_n^+ = k) y^n = \phi(k; y) .\end{aligned}$$

Because of $\sum_{r>0} \left(\frac{\rho v^2}{1+\rho v^2}\right)^r = \rho v^2$, we obtain

$$\phi(k; y) = \frac{1}{\gamma}(\alpha v^2 + \beta v + \gamma)(\rho v)^k \frac{1}{1-v}.$$

Extracting

$$[v^n] \left\{ (\alpha v^2 + \beta v + \gamma)^n (\rho v)^k \frac{1 - \rho v^2}{1 - v} \right\}$$

yields the following expression for $\mathbf{P}(\mathbf{D}_n^+ = k)$:

$$\begin{aligned}\mathbf{P}(\mathbf{D}_n^+ = k) &= \rho^k \sum_{j \ge 0} \left[\binom{n; \alpha, \beta, \gamma}{n - k - j} - \rho \binom{n; \alpha, \beta, \gamma}{n - k - 2 - j} \right] = \\ &= \rho^k \mathbf{P}(\mathbf{S}_n \le -k) - \rho^{k+1} \mathbf{P}(\mathbf{S}_n \le -k - 2)\end{aligned} \tag{6}$$

Alternatively, expression (6) can be rewritten as

$$\mathbf{P}(\mathbf{D}_n^+ = k) = \rho^k \left[\binom{n; \alpha, \beta, \gamma}{n - k} + \binom{n; \alpha, \beta, \gamma}{n - k - 1} + (1 - \rho) \sum_{j \ge 0} \binom{n; \alpha, \beta, \gamma}{n - k - 2 - j} \right], \tag{7}$$

which yields for the simplest special case $k = n$:

$$\mathbf{P}(\mathbf{D}_n^+ = n) = \rho^n \binom{n; \alpha, \beta, \gamma}{0} = \alpha^n.$$

Similarly, summation over $k \geq 0$ for fixed r in (4) yields the g.f. $\phi(r;y)$ for $\mathbf{P}(\mathbf{Q}_n = r)$:

$$\sum_k \phi(k,r;y) = \sum_k \sum_{n\geq 0} \mathbf{P}(\mathbf{D}_n^+ = k,\ \mathbf{Q}_n = r) y^n =$$
$$= \sum_{n\geq 0} \mathbf{P}(\mathbf{Q}_n = r) y^n = \phi(r;y)\,.$$

Because $\sum_{k\geq 0}(\rho v)^k = \frac{1}{1-\rho v}$, we get for $\phi(r;y)$

$$\phi(r;y) = \frac{1}{\gamma}\left(\frac{\rho v^2}{1+\rho v^2}\right)^r (\alpha v^2 + \beta v + \gamma)\frac{1}{\rho v^2}\frac{1}{1-\rho v}\frac{1}{1-v}.$$

Picking out the coefficient

$$[v^n]\left\{\frac{\rho^{r-1}v^{2r-2}}{(1+\rho v^2)^r}(\alpha v^2+\beta v+\gamma)^n \frac{1-\rho v^2}{1-\rho v}\frac{1}{1-v}\right\},$$

yields the expression for $\mathbf{P}(\mathbf{Q}_n = r)$:

$$\mathbf{P}(\mathbf{Q}_n = r) = \rho^{r-1}\sum_{i\geq 0}\rho^i\binom{-r}{i}\sum_{j\geq 0}\frac{1-\rho^{j+1}}{1-\rho}\left[\binom{n;\alpha,\beta,\gamma}{n-2r+2-2i-j} - \rho\binom{n;\alpha,\beta,\gamma}{n-2r-2i-j}\right], \tag{8}$$

which can be rewritten as

$$\mathbf{P}(\mathbf{Q}_n = r) = \rho^{r-1}\sum_{i\geq 0}\rho^i\binom{-r}{i}\times$$
$$\left[\binom{n;\alpha,\beta,\gamma}{n-2r+2-2i} + \sum_{j\geq 1}(1+\rho^j)\binom{n;\alpha,\beta,\gamma}{n-2r+2-2i-j}\right]. \tag{9}$$

For the simplest special case, $r = \frac{n}{2}+1$, and obviously n even, we get from (9)

$$\mathbf{P}(\mathbf{Q}_n = \frac{n}{2}+1) = \rho^{\frac{n+2}{2}-1}\binom{n;\alpha,\beta,\gamma}{0} = (\alpha\gamma)^{\frac{n}{2}}.$$

4. Moments. The interesting moments can be derived using appropriately defined moment generating functions (mgf.). Starting point is again the generating function for $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$:

$$\phi(k,r;y) = \frac{1}{\gamma}(\alpha v^2+\beta v+\gamma)\frac{1}{\rho v^2}\left(\frac{\rho v^2}{1+\rho v^2}\right)^r (\rho v)^k \frac{1}{1-v}.$$

Multiplication of $\phi(k,r;y)$ by e^{tk} and by e^{sr} und summation over all admissible values for k and r, i.e. $k \geq 0$ and $r > 0$, yields

$$\begin{aligned} G(t,s) &= \sum_k \sum_r \phi(k,r;y) e^{tk} e^{sr} \\ &= \sum_{n\geq 0} \sum_k \sum_r \mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) e^{tk} e^{sr} y^n \\ &= \sum_{n\geq 0} M_{D_n^+, Q_n}(t,s) y^n ; \end{aligned}$$

$G(t,s)$ is therefore the generating function for the bivariate mgf. and $[y^n]G(t,s)$ is the joint moment generating function $M_{D_n^+,Q_n}(t,s)$ of $(\mathbf{D}_n^+, \mathbf{Q}_n)$. The generating function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\gamma}(\alpha v^2 + \beta v + \gamma) \frac{e^s}{1-\rho v^2(e^s-1)} \frac{1}{1-\rho v e^t} \frac{1}{1-v},$$

and, technically, $M_{D_n^+,Q_n}(t,s)$ can be derived as

$$[v^n] \left\{ e^s (\alpha v^2 + \beta v + \gamma)^n \frac{1}{1-\rho v^2(e^s-1)} \frac{1}{1-\rho v e^t} \frac{1-\rho v^2}{1-v} \right\}.$$

Extracting the coefficient of v^n yields the bivariate mgf. of $(\mathbf{D}_n^+, \mathbf{Q}_n)$:

$$\begin{aligned} M_{D_n^+,Q_n}(t,s) = \frac{e^s}{1-\rho e^t} & \sum_{i\geq 0} \rho^i (e^s-1)^i \times \\ & \sum_{j\geq 0} (1-(\rho e^t)^{j+1}) \left[\binom{n;\alpha,\beta,\gamma}{n-2i-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-2i-j} \right]. \end{aligned} \tag{10}$$

The joint moment $\mathbf{E}(\mathbf{D}_n^+ \, \mathbf{Q}_n)$ can now be derived from (10) as $\frac{\partial^2}{\partial t \partial s} M_{D_n^+,Q_n}(t,s)|_{t=s=0}$ and is given by

$$\begin{aligned} \mathbf{E}(\mathbf{D}_n^+ \, \mathbf{Q}_n) = & \frac{\rho}{(1-\rho)^2} \times \\ & \sum_{j\geq 0} (j\rho^{j+1} - (j+1)\rho^j + 1) \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho^2 \binom{n;\alpha,\beta,\gamma}{n-4-j} \right]. \end{aligned} \tag{11}$$

The mgf's for the random variables $\mathbf{D}_n^+$ and $\mathbf{Q}_n$ can be obtained by substituting $s=0$ and $t=0$, respectively in the bivariate mgf. (10).

The substitution $s=0$ in (10) yields the mgf. $M_{D_n^+}(t)$ for $\mathbf{D}_n^+$:

$$M_{D_n^+}(t) = \frac{1}{1-\rho e^t} \sum_{j\geq 0} (1-(\rho e^t)^{j+1}) \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-j} \right],$$

which, after expanding ρe^t, can be rewritten as

$$M_{D_n^+}(t) = \sum_{m\geq 0} \frac{t^m}{m!} \sum_{j\geq 0} \sum_{i=0}^{j} i^m \rho^i \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-j} \right], \tag{12}$$

and the m-th moment of $\mathbf{D}_n^+$, given by $\left[\frac{t^m}{m!}\right] M_{D_n^+}(t)$, can immediately be seen from (12):

$$\mathbf{E}(\mathbf{D}_n^{+m}) = \sum_{j\geq 0} \sum_{i=0}^{j} i^m \rho^i \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-j} \right].$$

Specializing on $m = 1, 2$ we get for the first two moments of $\mathbf{D}_n^+$:

$$\mathbf{E}(\mathbf{D}_n^+) = \frac{\rho}{(1-\rho)^2} \sum_{j\geq 0} \left[j\rho^{j+1} - (j+1)\rho^j + 1 \right] \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-j} \right], \tag{13}$$

and

$$\begin{aligned} \mathbf{E}(\mathbf{D}_n^{+2}) = \frac{\rho}{(1-\rho)^3} \sum_{j\geq 0} & \left[-j^2\rho^{j+2} + (2j(j+1)-1)\rho^{j+1} - (j+1)^2\rho^j + \rho + 1 \right] \times \\ & \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-j} \right]. \end{aligned} \tag{14}$$

The substitution of $t = 0$ in (10) yields the mgf. $M_{Q_n}(s)$ of $\mathbf{Q}_n$:

$$M_{Q_n}(s) = e^s \sum_{i\geq 0} \rho^i (e^s - 1)^i \sum_{j\geq 0} \frac{1-\rho^{j+1}}{1-\rho} \left[\binom{n;\alpha,\beta,\gamma}{n-2i-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-2i-j} \right]. \tag{15}$$

The m-th moment of $\mathbf{Q}_n$ can now be derived as $\frac{d^m}{ds^m} M_{Q_n}(s)|_{s=0}$ and can be written as

$$\mathbf{E}(\mathbf{Q}_n^m) = \sum_{i\geq 0} \rho^i f_i^{(m)}(0) \sum_{j\geq 0} S_j(i),$$

where

$$f_i^{(m)}(0) := \frac{d^m}{ds^m} e^s (e^s - 1)^i |_{s=0} = \sum_{\nu\geq 0} (-1)^\nu \binom{i}{\nu} (i - \nu + 1)^m,$$

and

$$S_j(i) = \frac{1-\rho^{j+1}}{1-\rho} \left[\binom{n;\alpha,\beta,\gamma}{n-2i-j} - \rho \binom{n;\alpha,\beta,\gamma}{n-2-2i-j} \right].$$

Specializing on $m = 1, 2$ we get for the first two moments of $\mathbf{Q}_n$:

$$\mathbf{E}(\mathbf{Q}_n) = \sum_{j\geq 0} \frac{1-\rho^{j+1}}{1-\rho} \left[\binom{n;\alpha,\beta,\gamma}{n-j} - \rho^2 \binom{n;\alpha,\beta,\gamma}{n-4-j} \right], \tag{16}$$

and

$$\mathbf{E}(\mathbf{Q}_n^2) = \sum_{j\geq 0} \frac{1-\rho^{j+1}}{1-\rho} \left[\binom{n;\alpha,\beta,\gamma}{n-j} + 2\rho \binom{n;\alpha,\beta,\gamma}{n-2-j} - \rho^2 \binom{n;\alpha,\beta,\gamma}{n-4-j} - 2\rho^3 \binom{n;\alpha,\beta,\gamma}{n-6-j} \right]. \tag{17}$$

5. Special Cases. In this section we will collect the pertaining results for two special cases: (i) for the symmetric random walk, i.e. a simple random walk with $\alpha = \gamma$ and therefore $\rho = 1$, and (ii) for the purely binomial random walk with $\alpha = \gamma = \frac{1}{2}$ and $\beta = 0$. Obviously, all results given in this section can be obtained by specialization of the corresponding results for the simple random walk.

The Symmetric Random Walk. For the symmetric random walk we immediately obtain from (5) for the joint probability $\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r)$

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) = \sum_{i\geq 0} \binom{-r}{i} \left[\binom{n;\alpha,\beta,\alpha}{A+2-2i} + \binom{n;\alpha,\beta,\alpha}{A+1-2i} \right], \tag{18}$$

where again $A = n - 2r - k$. For the marginal distributions we get from (7) and (9), respectively:

$$\mathbf{P}(\mathbf{D}_n^+ = k) = \binom{n;\alpha,\beta,\alpha}{n-k} + \binom{n;\alpha,\beta,\alpha}{n-k-1} = \mathbf{P}(\mathbf{S}_n = -k) + \mathbf{P}(\mathbf{S}_n = -k-1), \tag{19}$$

and

$$\mathbf{P}(\mathbf{Q}_n = r) = \sum_{i\geq 0} \binom{-r}{i} \left[\binom{n;\alpha,\beta,\alpha}{n-2r+2-2i} + 2\sum_{j\geq 1} \binom{n;\alpha,\beta,\alpha}{n-2r+2-2i-j} \right]. \tag{20}$$

For the joint moment $\mathbf{E}(\mathbf{D}_n^+ \mathbf{Q}_n)$ we obtain from (11)

$$\mathbf{E}(\mathbf{D}_n^+ \mathbf{Q}_n) = \sum_{j\geq 0} \binom{j+1}{2} \left[\binom{n;\alpha,\beta,\alpha}{n-j} - \binom{n;\alpha,\beta,\alpha}{n-4-j} \right], \tag{21}$$

whereas the first two moments of $\mathbf{D}_n^+$ and $\mathbf{Q}_n$ are given by

$$\begin{aligned} \mathbf{E}(\mathbf{D}_n^+) &= \sum_{j\geq 1} [2j-1] \binom{n;\alpha,\beta,\alpha}{n-j}, \\ \mathbf{E}(\mathbf{D}_n^{+2}) &= \sum_{j\geq 1} [2j(j-1)+1] \binom{n;\alpha,\beta,\alpha}{n-j}, \end{aligned} \tag{22}$$

and

$$
\begin{aligned}
\mathbf{E}(\mathbf{Q}_n\ ) &= \binom{n;\alpha,\beta,\alpha}{n} + 2\binom{n;\alpha,\beta,\alpha}{n-1} + 3\binom{n;\alpha,\beta,\alpha}{n-2} + 4\sum_{j\geq 0}\binom{n;\alpha,\beta,\alpha}{n-3-j}, \\
\mathbf{E}(\mathbf{Q}_n{}^2) &= \binom{n;\alpha,\beta,\alpha}{n} + 2\binom{n;\alpha,\beta,\alpha}{n-1} + 5\binom{n;\alpha,\beta,\alpha}{n-2} + 8\binom{n;\alpha,\beta,\alpha}{n-3} + 10\binom{n;\alpha,\beta,\alpha}{n-4} + \\
&\quad 12\sum_{j\geq 0}\binom{n;\alpha,\beta,\alpha}{n-5-j},
\end{aligned}
\tag{23}
$$

respectively, which follow immediately from formulae (13), (14), (16), and (17).

The purely binomial random walk. The results for the purely binomial random walk can easily be obtained by specialization of the results for the symmetric random walk. Technical prerequistes are the relation

$$\binom{n;1/2,0,1/2}{2m} = 2^{-n}\binom{n}{m}, \tag{24}$$

cf. Panny (1984), and the following formulae, which allow the simplification of some expressions, cf. Riordan (1968, p.34):

$$
\begin{aligned}
\sum_{j\geq 0} j\binom{2\alpha}{\alpha-j} &= \alpha\binom{2\alpha-1}{\alpha}, \\
\sum_{j\geq 0} j\binom{2\alpha+1}{\alpha-j} &= (2\alpha+1)\binom{2\alpha-1}{\alpha} - 2^{2\alpha-1}, \quad \alpha = 1,2,\dots,
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
\sum_{j\geq 0} j(j-1)\binom{2\alpha}{\alpha-j} &= \alpha 2^{2\alpha-2} - \alpha\binom{2\alpha-1}{\alpha}, \quad \alpha = 2,3,\dots, \\
\sum_{j\geq 0} j(j-1)\binom{2\alpha+1}{\alpha-j} &= (\alpha+2)2^{2\alpha-1} - (\alpha+1)\binom{2\alpha+1}{\alpha}.
\end{aligned}
\tag{26}
$$

In the following we will collect the results for the purely binomial random walk:

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) = 2^{-n}\binom{n-r}{\lfloor\frac{n-2r-k+2}{2}\rfloor}, \tag{27}$$

which follows from (18) and an application of Vandermonde's convolution formula. For $\mathbf{D}_n^+$ we obtain from (19)

$$\mathbf{P}(\mathbf{D}_n^+ = k) = 2^{-n}\binom{n}{\lfloor\frac{n-k}{2}\rfloor}, \tag{28}$$

cf. also Revesz (1990, Theorem 2.4, p. 14). Because of

$$\mathbf{P}(\mathbf{D}_{n-r}^+ = r+k-2) = 2^{-(n-r)}\binom{n-r}{\lfloor\frac{n-r-(r+k-2)}{2}\rfloor},$$

we get immediately the following relation

$$\mathbf{P}(\mathbf{D}_n^+ = k, \mathbf{Q}_n = r) = 2^{-r}\mathbf{P}(\mathbf{D}_{n-r}^+ = r + k - 2). \tag{29}$$

For the joint moment we obtain after specializing of formula (21) and an application of (25):

$$\mathbf{E}(\mathbf{D}_n^+ \mathbf{Q}_n) = \begin{cases} \frac{2(n+1)(n+3)}{(n+2)} 2^{-n} \binom{n}{\frac{n}{2}} - 3, & n \text{ even}, \\ (2n+5)\, 2^{-n} \binom{n}{\frac{n-1}{2}} - 3, & n \text{ odd}. \end{cases} \tag{30}$$

For the the first two moments of $\mathbf{D}_n^+$ we get after simplification of formula (22) and an application of (25) and (26), respectively

$$\mathbf{E}(\mathbf{D}_n^+) = \begin{cases} (n+\frac{1}{2})\, 2^{-n} \binom{n}{\frac{n}{2}} - \frac{1}{2}, & n \quad \text{even}, \\ (n+1)\, 2^{-n} \binom{n}{\frac{n-1}{2}} - \frac{1}{2}, & n \quad \text{odd}, \end{cases} \tag{31}$$

and

$$\mathbf{E}(\mathbf{D}_n^{+2}) = \begin{cases} (n+\frac{1}{2}) - (n+\frac{1}{2})2^{-n} \binom{n}{\frac{n}{2}}, & n \quad \text{even}, \\ (n+\frac{1}{2}) - (n+1)2^{-n} \binom{n}{\frac{n-1}{2}}, & n \quad \text{odd}. \end{cases} \tag{32}$$

For $\mathbf{P}(\mathbf{Q}_n = r)$ we obtain from (20) and repeated applications of Vandermonde's convolution formula

$$\mathbf{P}(\mathbf{Q}_n = r) = 2^{-n} \sum_{j \geq 0} \binom{n-r}{\lfloor \frac{n-2r+2-j}{2} \rfloor}. \tag{33}$$

Using formula (28) it is easily seen that

$$\mathbf{P}(\mathbf{Q}_n = r) = 2^{-r}\mathbf{P}(\mathbf{D}_{n-r}^+ \geq r - 2),$$

which was originally proved by Csaki, cf. Revesz (1990, p.157). For the special case $r = 1$ we obtain from (33)

$$\mathbf{P}(\mathbf{Q}_n = 1) = \begin{cases} \frac{1}{2} + 2^{-n} \binom{n-1}{\frac{n}{2}}, n & \text{even}, \\ \frac{1}{2} + 2^{-n} \binom{n-1}{\frac{n-1}{2}}, n & \text{odd}. \end{cases}$$

Finally, for the first two moments of $\mathbf{Q}_n$ we get after specialization of (23)

$$\mathbf{E}(\mathbf{Q}_n) = \begin{cases} 2 - \frac{2(n+1)}{n+2} 2^{-n} \binom{n}{\frac{n}{2}}, & n \text{ even}, \\ 2 - 2 \cdot 2^{-n} \binom{n}{\frac{n-1}{2}}, & n \text{ odd}, \end{cases} \tag{34}$$

and

$$\mathbf{E}(\mathbf{Q}_n^2) = \begin{cases} 6 - \frac{2(n+1)(7n+20)}{(n+2)(n+4)} 2^{-n} \binom{n}{\frac{n}{2}}, & n \text{ even}, \\ 6 - \frac{2(7n+13)}{n+3} 2^{-n} \binom{n}{\frac{n-1}{2}}, & n \text{ odd}. \end{cases} \tag{35}$$

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